LECTURE 15: FIBER BUNDLES

1. Fiber Bundles

Recall that a rank k vector bundle is triple (π, E, M), where E is a smooth manifold called the total space, M a smooth manifold called the base, and $\pi : E \to M$ a surjective smooth map called the bundle projection map, so that one can find an open covering $\{U_\alpha\}$ of M and for each α a diffeomorphisms $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$, such that

(1) For each $m \in U_\alpha$, $\Phi_\alpha(\pi^{-1}(m)) = \{m\} \times \mathbb{R}^k$,
(2) If $m \in U_\alpha \cap U_\beta$, the transition map

$$g_{\beta\alpha}(m) = \Phi_\beta \circ \Phi_\alpha^{-1} : \{m\} \times \mathbb{R}^k \to \{m\} \times \mathbb{R}^k$$

is linear, and smoothly depends on $m \in U \cap V$.

Note that the transition functions $g_{\beta\alpha}$ are linear and invertible with inverse $g_{\alpha\beta}$, i.e. they define a map from $U_\alpha \cap U_\beta$ to the general linear group $\mathrm{GL}(k, \mathbb{R})$. So the condition (2) can be equally replaced by

(2′) For $U_\alpha \cap U_\beta \neq \emptyset$, there is a smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ so that

$$\Phi_\beta \circ \Phi_\alpha^{-1}(m, v) = (m, g_{\beta\alpha}(m)(v)), \quad \forall m \in U_\alpha \cap U_\beta, v \in \mathbb{R}^k.$$

In applications, especially in differential geometry and in mathematical physics, one need to study bundles whose fibers are no longer vector spaces, and thus the transition maps are no longer linear. This motivates the following definition

Definition 1.1. Let F be a smooth manifold, G a Lie group acting smoothly and effectively[1] on F. A *fiber bundle* with *fiber* F and *structural group* G is a triple (π, E, M), where E and M are smooth manifolds and $\pi : E \to M$ a surjective smooth map, so that one can find an open covering $\{U_\alpha\}$ of M and for each α a diffeomorphisms $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$, such that

(1) For each $m \in U_\alpha$, $\Phi_\alpha(\pi^{-1}(m)) = \{m\} \times F$,
(2) For $U_\alpha \cap U_\beta \neq \emptyset$, there is a smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ so that

$$\Phi_\beta \circ \Phi_\alpha^{-1}(m, v) = (m, g_{\beta\alpha}(m)(v)), \quad \forall m \in U_\alpha \cap U_\beta, v \in F.$$

Again E is called the total space, M is called the base, π is called the bundle projection map, Φ_α is called the local trivialization map, $g_{\beta\alpha}$ is called the transition map, and $E_m = \pi^{-1}(m)$ is called the fiber over the point m.

[1] We say an action τ of G on M is *effective* if each element in G does acts on M, i.e. $\tau_g \neq \mathrm{Id}$ for all $g \neq e$. Any Lie group action can be converted into an effective action by replacing G with $G/\ker(\tau)$.

As in the case of vector bundles, a section over an open set $U \subset M$ is a smooth map $s : U \to E$ so that $\pi \circ s = \mathrm{Id}_U$. The set of all smooth section over U is denoted by $\Gamma^\infty(U, E)$. However, we no longer have the conception of (local) frame.

There are two important special cases:

- If the fiber is $\mathbb{R}^k$, and the structural group is $\mathrm{GL}(k, \mathbb{R})$, then we get rank k vector bundle.
- If the fiber F is the structural group G, acting on itself by left translations, then we say the fiber bundle is a *principal G-bundle*.

Example. (**The Hopf fibration**) Consider the complex projective space, i.e. the set of all complex lines inside $\mathbb{C}^{n+1}$,

$$\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\}/\sim .$$

As in the case of $\mathbb{RP}^n$, one can prove $\mathbb{CP}^n$ is a smooth manifold, with coordinate charts

$$U_j = \{[z^1 : z^2 : \cdots : z^{n+1}] \mid z_j \neq 0\} \xrightarrow{\varphi_j} \mathbb{C}^n = \{(\frac{z^1}{z^j}, \cdots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \cdots, \frac{z^{n+1}}{z^j})\}.$$

Now view $S^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ as the set of all $(z^1, \cdots, z^{n+1}) \in \mathbb{C}^{n+1}$ with $|z^1|^2 + \cdots + |z^{n+1}|^2 = 1$. Then the projection map $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$ restricts to a surjective smooth map

$$\pi : S^{2n+1} \to \mathbb{CP}^n.$$

We claim that this makes S^{2n+1} into a fiber bundle over $\mathbb{CP}^n$ with fiber S^1. This is known as the *Hopf fibration*.

In fact, by definition we have

$$\pi^{-1}(U_j) = \{z = (z^1, \cdots, z^{n+1}) \in S^{2n+1} \mid z^j \neq 0\}$$

and the local trivialization map can be taken as

$$\Phi_j : \pi^{-1}(U_j) \to U_j \times S^1, \quad \Phi_j(z) = ([z^1 : \cdots : z^{n+1}], \frac{z^j}{|z^j|}).$$

It is a diffeomorphism since it is a smooth map with smooth inverse

$$\Phi_j^{-1}([z^1 : \cdots : z^{n+1}], e^{i\theta}) = \frac{e^{i\theta}|z_j|}{\sqrt{|z^1|^2 + \cdots + |z^{n+1}|^2} z^j}(z^1, \cdots, z^{n+1}).$$

Finally for $j \neq k$, one has

$$\Phi_k \Phi_j^{-1}([z^1 : \cdots : z^{n+1}], e^{i\theta}) = ([z^1 : \cdots : z^{n+1}], e^{i\theta}\frac{z^k|z^j|}{z^j|z^k|}).$$

So the transition map is

$$g_{kj} : U_k \cap U_l \to S^1, \quad [z^1 : \cdots : z^{n+1}] \mapsto \frac{z^k|z^j|}{z^j|z^k|}.$$

So S^{2n+1} is in fact a principal S^1-bundle over $\mathbb{CP}^n$.

Now suppose (π, E, M) is a fiber bundle. So there is an open covering $\{U_\alpha\}$ of M and transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$. By definition, these transition maps satisfy the following *cocycle condition*:

$$g_{\gamma\alpha}(m) = g_{\gamma\beta}(m) g_{\beta\alpha}(m), \qquad \forall m \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Conversely, we have

Proposition 1.2. *Let G be a Lie group and $\{U_\alpha\}$ be an open covering of M. Suppose there is a collection of maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ satisfying the cocycle condition*

$$g_{\gamma\alpha}(m) = g_{\gamma\beta}(m) g_{\beta\alpha}(m), \qquad \forall m \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Then for any smooth manifold F on which there is an effective G-action, there exists a fiber bundle (π, E, M) over M with fiber F, structural group G, so that whose transition maps are exactly given by $g_{\beta\alpha}$'s.

Proof. Consider the disjoint union $\mathcal{E} = \bigcup_\alpha U_\alpha \times F$. Define an equivalence relation on $\mathcal{E}$ by

$$(m, v_1) \sim (m, v_2) \Longleftrightarrow v_2 = g_{\beta\alpha}(m)(v_1) \text{ for some } \alpha, \beta.$$

Let E be the quotient space and denote by $pr : \mathcal{E} \to E$ the quotient map. Note that $pr : \mathcal{E} \to E$ maps each $U_\alpha \times F$ homeomorphically to the open set $pr(U_\alpha \times F)$ in E. Thus E is a smooth manifold with local chars $\{U_\alpha \times V_\beta\}$, where $\{V_\beta\}$ is a local chart of F. Moreover, if we let $\pi : E \to M$ be the obvious projection map, then $\pi^{-1}(U_\alpha) = pr(U_\alpha \times F)$, and E is a fiber bundle over M with local trivialization the inverse of $pr|_{U_\alpha \times F}$. Also by construction, the transition maps are exactly the $g_{\beta\alpha}$'s. $\square$

Note that by this construction, in particular we can get many vector bundles or principal bundles.

Example. (**The Möbius band**) Consider the action of $G = \{\pm 1\}$ on $\mathbb{R}$ by multiplication. Let $M = S^1$, with open cover $\{U_+, U_-\}$, where $U_+ = S^1 - \{(0, -1)\}$ and $U_- = S^1 - \{(0, 1)\}$. Define a transition map

$$g_{-+} : U_+ \cap U_- \to G, \quad (a, b) \mapsto \operatorname{sgn}(a).$$

Then by the construction above, we get a rank 1 vector bundle over S^1. This is known as the *Möbius band.*

2. Principal Bundles

Usually we will denote the total space of a principal bundle by P. Now let (π, P, M) be a principal G-bundle. Then one gets a natural *right* G-action on P as follows: For $g \in G$ and $p \in P$. Take α so that $\pi(p) \in U_\alpha$. Then $\Phi_\alpha(p) = (\pi(p), g_\alpha)$ for some $g_\alpha \in G$. Define

$$p \cdot g = \Phi_\alpha^{-1}(\pi(p), g_\alpha g) \in \pi^{-1}(\pi(p)).$$

This definition is independent of choices of U_α: If we also have $\Phi_\beta(p) = (\pi(p), g_\beta)$ for some $g_\beta \in G$, then $g_{\beta\alpha}(\pi(p)) = g_\beta g_\alpha^{-1}$. Thus $\Phi_\alpha^{-1}(\pi(p), g_\alpha g) = \Phi_\beta^{-1}(\pi(p), g_\beta g)$. It is also easy to check that we do get a right G action, and that the action is free and proper. As a result, the base M can be identified with the orbit space P/G. [In other words, the quotient map $\pi : M \to M/G$ in theorem 2.4 of lecture 14 is not only a submerison, but makes M into a principal-G bundle over M/G.]

Remark. We know any vector bundle admits at least one global section: the zero section. This fact is no longer true for fiber bundles. In fact, if a principle G-bundle E admits a global section $s : M \to E$, then E has to be a trivial bundle $M \times G$, since

$$\Phi : E \to M \times G, \quad p \mapsto (\pi(p), g)$$

where $g \in G$ is the element so that $p \cdot g = s(\pi(p))$, gives us a global trivialization.

Note that by proposition 1.2, for any fiber bundle over M with fiber F and structure group G, one can construct a principle G-bundle over M whose transition maps coincide with the transition maps of the given fiber bundle. Conversely, proposition 1.2 also tells us that one can construct fiber bundle from the principal bundle. However, the right G-action on P allows us to do so more explicitly: Let (π_P, P, M) be a principle bundle, F a smooth manifold admitting an effective left G-action. Then we define an equivalence relation on $P \times F$ via

$$(p, v) \sim (p \cdot g, g^{-1} \cdot v).$$

The quotient space is denoted by

$$P \times_G F = P \times F / \sim .$$

One can define a projection map $\pi : P \times_G F \to M$ by

$$\pi([p, v]) = \pi_P(p).$$

This is a surjective smooth map. One can check that $(\pi, P \times_G F, M)$ is then a fiber bundle with fiber F and structure group G, and that its transition maps coincide with those of (π_P, P, M).

Example. (**The frame bundle**) Let (π, E, M) be a rank k vector bundle over M. Since $\mathrm{GL}(k, \mathbb{R})$ acts effectively on $\mathbb{R}^k$, we can get a principle $\mathrm{GL}(k, \mathbb{R})$-bundle, $F(E)$, over M. This is called the *frame bundle* of E. The fiber over m of the frame bundle $F(E)$ is given by

$$F(E)_m = \{(v_1, \cdots, v_k) \mid v_1, \cdots, v_k \text{ form an basis of } E_m\},$$

which is diffeomorphic to $\mathrm{GL}(k, \mathbb{R})$.

Example. (**The associated vector bundle**) Let P be a principle G-bundle over M, where G is a linear group, and let W be a k dimensional vector space with a linear G-action. Then $P \times_G W$ is a rank k vector bundle over M.

Remark. If E is a rank k vector bundle over M, then $E = F(E) \times_{GL(k,\mathbb{R})} \mathbb{R}^k$.