LECTURE 18: INTEGRATION ON MANIFOLDS

1. Orientations and Integration

Before we integrate an n-form on manifold, we need an orientation.

Definition 1.1. Let M be a smooth manifold of dimension n.

(1) Two charts $(\varphi_\alpha, U_\alpha, V_\alpha)$ and $(\varphi_\beta, U_\beta, V_\beta)$ are *orientation compatible* if the transition map $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfies
$$\det(d\varphi_{\alpha\beta})_x > 0$$
for all $x \in \varphi_\alpha(U_\alpha \cap U_\beta)$.

(2) An *orientation* of M is an atlas $\mathcal{A} = \{(\varphi_\alpha, U_\alpha, V_\alpha) \mid \alpha \in \Lambda\}$ whose charts are pairwise orientation compatible. We say M is *orientable* if it has an orientation.

Example. If M admits an atlas that consists of only one chart, then M is automatically oriented. In particular, any "graph manifold" is oriented.

Example. If M admits an atlas that consists of two charts, and the two charts are not orientation compatible, then one can make a new atlas by keeping the first chart, and replace the second chart map by composing it with the map $(x^1, x^2, \cdots, x^n) \to (-x^1, x^2, \cdots, x^n)$. Then the two charts in the new atlas are orientation compatible. So M is orientable. As a consequence, for any n, S^n is orientable.

Example. For the real projective space $\mathbb{RP}^n$, we have constructed an atlas consisting of $n+1$ charts. We have calculated the transition map $\varphi_{1,n+1}$ and got
$$\varphi_{1,n+1}(y^1, \cdots, y^n) = \left(\frac{1}{y^n}, \frac{y^1}{y^n}, \cdots, \frac{y^{n-1}}{y^n}\right).$$
A simple computation yields $\det(d\varphi_{1,n+1}) = (-1)^{n+1}\frac{1}{(y^n)^{n+1}}$. It follows that for the atlas we constructed, (φ_1, U_1, V_1) and $(\varphi_{n+1}, U_{n+1}, V_{n+1})$ are orientation compatible if and only if n is odd. One can do the same computation for other pairs of charts. In fact, it is true that $\mathbb{RP}^n$ is orientable if and only if n is odd.

Remark. We say two orientations $\mathcal{A}$ and $\mathcal{B}$ are the same if $\mathcal{A} \cup \mathcal{B}$ is an orientation. Under this sense, it is obvious that if M is connected and orientable, then M admits exactly two *different* orientations.

Theorem 1.2. *An n-dimensional manifold M is orientable if and only if M admits a nowhere vanishing n-form ω.*

Proof. First let ω be a nowhere vanishing n-form on M. Then for each local chart $(U, x^1, \cdots, x^n)$, there is a function $f \neq 0$ so that $\omega = f dx^1 \wedge \cdots \wedge dx^n$. It follows that

$$\omega(\partial_1, \cdots, \partial_n) = f \neq 0.$$

We can always take such a chart near each point so that $f(x) > 0$, otherwise we can replace x^1 by $-x^1$. Now suppose $(U_\alpha, x_\alpha^1, \cdots, x_\alpha^n)$ and $(U_\beta, x_\beta^1, \cdots, x_\beta^n)$ be two such charts, so that $\omega = f dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = g dx_\beta^1 \wedge \cdots \wedge dx_\beta^n$, where $f, g > 0$. Then

$$0 < g = \omega(\partial_1^\beta, \cdots, \partial_n^\beta) = (\det d\varphi_{\alpha\beta})\omega(\partial_1^\alpha, \cdots, \partial_\alpha^n) = (\det d\varphi_{\alpha\beta}) f.$$

It follows that $\det(d\varphi_{\alpha\beta}) > 0$. So the atlas constructed by this way is an orientation.

Conversely, suppose $\mathcal{A}$ is an orientation. For each local chart U_α, we let $\omega_\alpha = dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$. Pick a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. We claim that $\omega = \sum_\alpha \rho_\alpha \omega_\alpha$ is a nowhere vanishing n-form on M. In fact, for each $p \in M$, there is a neighborhood U of p so that the sum $\sum_\alpha \rho_\alpha \omega_\alpha$ is a finite sum $\sum_{i=1}^k \rho_i \omega_i$. It follows that near p,

$$\omega(\partial_1^1, \cdots, \partial_n^1) = \sum_i (\det d\varphi_{1k}) \rho_i > 0.$$

So $\omega \neq 0$ near p. □

Remark. According to the proposition 1.5 in lecture 9, M admits a nowhere vanishing n-form if and only if the top form bundle $\wedge^n T^*M$ is a trivial line bundle.

Definition 1.3. A nowhere vanishing n-form on an n-dim manifold is called a *volume form.*

Note that if M is orientable, and ω is a volume form, then the two orientations of M corresponds to ω and $-\omega$ respectively.

In what follows we will always assume that M is oriented and fix an orientation on M. To define the integrals of n-forms ω on M, we first assume that ω is a n-form supported in an orientation-compatible coordinate chart $\{\varphi, U, V\}$, so that there is a function $f(x^1, \cdots, x^n)$ supported in U such that

$$\omega = f(x^1, \cdots, x^n) dx^1 \wedge \cdots \wedge dx^n.$$

We define

$$\int_U \omega := \int_V f(x^1, \cdots, x^n) dx^1 \cdots dx^n, \tag{1}$$

where the right hand side is the Lebesgue integral on $V \subset \mathbb{R}^n$.

To integrate a general n-form ω on M, we take a locally finite cover $\{U_\alpha\}$ of M that consists of orientation-compatible coordinate charts. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Now since each ρ_α is supported in U_α, each $\rho_\alpha \omega$ is supported U_α also. We define

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega. \tag{2}$$

We say that ω is *integrable* if the right hand side converges.

One need to check that the definition above is independent of the choice of orientation-compatible coordinate charts, and is independent of the choice of partition of unity, so that the integral is well-defined.

Theorem 1.4. *The expression (2) is independent of the choice of* $\{U_\alpha\}$ *and the choice of* $\{\rho_\alpha\}$.

Proof. We first show that (1) is well-defined, i.e. if ω is supported in U, and if $\{x^i_\alpha\}$ and $\{x^i_\beta\}$ are two orientation-compatible coordinate systems on U, so that

$$\omega = f_\alpha dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha = f_\beta dx^1_\beta \wedge \cdots \wedge dx^n_\beta,$$

then

$$\int_{V_\alpha} f_\alpha dx^1_\alpha \cdots dx^n_\alpha = \int_{V_\beta} f_\beta dx^1_\beta \cdots dx^n_\beta.$$

This is true, because

$$dx^1_\beta \wedge \cdots \wedge dx^n_\beta = \det(d\varphi_{\alpha\beta}) dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$$

implies $f_\alpha = \det(d\varphi_{\alpha\beta}) f_\beta$. On the other hand side, the change of variable formula in $\mathbb{R}^n$ reads

$$\int_{V_\beta} f dx^1_\beta \cdots dx^n_\beta = \int_{V_\alpha} f |\det(d\varphi_{\alpha\beta})| dx^1_\alpha \cdots dx^n_\alpha.$$

So the desired formula follows form the fact $\det(d\varphi_{\alpha\beta}) > 0$ since U_α and U_β are orientation-compatible.

To prove (2) is well-defined, we suppose $\{U_\alpha\}$ and $\{U_\beta\}$ are two locally finite cover of M consisting of orientation-compatible charts, and $\{\rho_\alpha\}$ and $\{\rho_\beta\}$ are partitions of unity subordinate to $\{U_\alpha\}$ and $\{U_\beta\}$ respectively. We consider a new cover $\{U_\alpha \cap U_\beta\}$ with new partition of unity $\{\rho_\alpha \rho_\beta\}$. It is enough to prove

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_{\alpha,\beta} \int_{U_\alpha \cap U_\beta} \rho_\alpha \rho_\beta \omega.$$

This is true because for each fixed α,

$$\int_{U_\alpha} \rho_\alpha \omega = \int_{U_\alpha} (\sum_\beta \rho_\beta) \rho_\alpha \omega = \sum_\beta \int_{U_\alpha \cap U_\beta} \rho_\beta \rho_\alpha \omega.$$

□

Obviously the integral defined above is linear:

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

Now suppose M, N are both oriented manifolds, with volume forms η_1, η_2 respectively.

Definition 1.5. A smooth map $f : M \to N$ is said to be *orientation-preserving* if $f^*\eta_2$ is a volume form on M that defines the same orientation as η_1 does.

Theorem 1.6. *Suppose $\varphi : M \to N$ is an orientation-preserving diffeomorphism, then*

$$\int_M f^*\omega = \int_N \omega.$$

Proof. It is enough to prove this in local charts, in which case this is merely the change of variable formula in $\mathbb{R}^n$. $\square$

Remark. If ω is a k-form on M, where $k < n = \dim M$, then one cannot integrate ω over M. However, for any k-dimensional orientable submanifold $X \subset M$, one can define $\int_X \omega$ by setting it to be $\int_X \iota^*\omega$, where $\iota : X \hookrightarrow M$ is the inclusion map.

Remark. If M is oriented and we fix a volume form η on M, then we can define the integral of a smooth function f on M to be $\int_M f\eta$.

Remark. If M is not oriented, one cannot define integral of differential forms as above. However, we can still integrate via *densities*.

2. The Stokes formula

The Stokes formula is one of the most important formula in calculus. We now extend it to smooth manifolds. We need some knowledge of manifolds with boundary. Denote

$$\mathbb{R}^n_+ = \{(x^1, \cdots, x^n) \mid x^n \geq 0\}.$$

Definition 2.1. A topological space M is called an n-dimensional *manifold with boundary* if it is Hausdorff, second-countable and such that for any $p \in M$, there is a neighborhood U of p which is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

Let M be a manifold with boundary, then we can define its boundary to be

$$\partial M = \{p \in M \mid p \text{ has no neighborhood that is homeomorphic to } \mathbb{R}^n\}.$$

As in the ordinary case, one can define a smooth structure on a manifold with boundary to be an atlas so that the transition map between each pair of charts is smooth.

Example. The closed ball $B^n(1) = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a smooth manifold with boundary, and its boundary is $\partial B^n(1) = S^{n-1}$.

Lemma 2.2. *Suppose M is a smooth manifold with boundary, then ∂M is a smooth manifold.*

Proof. Let $p \in \partial M$ and $(U, x^1, \cdots, x^n)$ be a coordinate chart near p that is homeomorphic to $\mathbb{R}^n_+$. Then $U \cap \partial M = \{(x^1, \cdots, x^n) \mid x^n = 0\}$. In other words, U is an adapted coordinate chart for ∂M. So ∂M is a smooth submanifold of M, with coordinate charts $(U \cap \partial M, x^1, \cdots, x^{n-1})$. $\square$

Theorem 2.3. *If M is an orientable manifold with boundary of dimension n, then the boundary ∂M is an orientable $n-1$ dimensional submanifold of M.*

Proof. Let $(U_\alpha, x^1_\alpha, \cdots, x^n_\alpha)$ and $(U_\beta, x^1_\beta, \cdots, x^n_\beta)$ be two orientation compatible charts of M near $p \in \partial M$ so that $M \cap U_\alpha$ is characterized by $x^n_\alpha \geq 0$, and $M \cap U_\beta$ is characterized by $x^n_\beta \geq 0$. We would like to show that the coordinate charts $(U_\alpha \cap \partial M, x^1_\alpha, \cdots, x^{n-1}_\alpha)$ and $(U_\beta \cap \partial M, x^1_\beta, \cdots, x^{n-1}_\beta)$ are orientation compatible. In fact, if we denote the transition map $\varphi_{\alpha\beta}$ between U_α and U_β by $(\varphi^1, \cdots, \varphi^n)$, then on $\partial M \cap U_\alpha \cap U_\beta$, $x^n_\alpha = x^n_\beta = 0$. In other words,

$$\varphi^n(x^1, \cdots, x^{n-1}, 0) = 0$$

on $(U_\alpha \cap \partial M) \cap (U_\beta \cap \partial M)$. It follows

$$\frac{\partial \varphi^n}{\partial x^i}(x^1, \cdots, x^{n-1}, 0) = 0, \quad i = 1, \cdots, n-1.$$

Moreover, the relation $\varphi^n(x^1, \cdots, x^n) > 0$ for $x^n > 0$ implies

$$\frac{\partial \varphi^n}{\partial x^n}(x^1, \cdots, x^{n-1}, 0) > 0.$$

Since $(U_\alpha, x^1_\alpha, \cdots, x^n_\alpha)$ and $(U_\beta, x^1_\beta, \cdots, x^n_\beta)$ are orientation compatible, $\det(\frac{\partial \varphi^i}{\partial x^j}) > 0$. In particular,

$$\det\left(\frac{\partial \varphi^i}{\partial x^j}(x^1, \cdots, x^{n-1}, 0)\right) = \det\begin{pmatrix} \left(\frac{\partial \varphi^i}{\partial x^j}(x^1, \cdots, x^{n-1}, 0)\right)_{1 \leq i,j \leq n-1} & * \\ 0 & \frac{\partial \varphi^n}{\partial x^n}(x^1, \cdots, x^{n-1}, 0) \end{pmatrix} > 0.$$

It follows that

$$\det\left(\frac{\partial \varphi^i}{\partial x^j}(x^1, \cdots, x^{n-1}, 0)\right)_{1 \leq i,j \leq n-1} > 0.$$

This proves that the charts $(U_\alpha \cap \partial M, x^1_\alpha, \cdots, x^{n-1}_\alpha)$ and $(U_\beta \cap \partial M, x^1_\beta, \cdots, x^{n-1}_\beta)$ of ∂M are orientation compatible. □

Remark. The boundary of a non-orientable manifold could be oriented (e.g. the Mobiüs band) or non-oriented (e.g. $[0, 1] \times M$, where M is non-orientable).

Remark. According to Pontrjagin and Thom, a closed manifold is the boundary of another manifold if and only if specific Stiefel-Whitney numbers are zero. For example, $\mathbb{CP}^{2k}$ is not the boundary of any other manifolds.

The *induced orientation* on ∂M is defined as follows: Suppose locally near boundary M is described as $x^n \geq 0$, and that the orientiation of M is described by $dx^1 \wedge \cdots \wedge dx^n$. Then locally the positive orientation on ∂M is the one that corresponds to the differential form

$$(-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}.$$

Note that $-x^n$ is the "out normal direction", and this boundary orientation is chosen so that

$$d(-x^n) \wedge (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1} = dx^1 \wedge \cdots \wedge dx^n.$$

Finally we can state

Theorem 2.4 (Stokes' theorem)**.** *Let M be a compact oriented n-dimensional manifold with boundary ∂M (with the induced orientation above). For any $\omega \in \Omega^{n-1}(M)$,*

$$\int_{\partial M} \omega = \int_M d\omega.$$

Proof. We have three cases:

(1) ω is supported in U that is diffeomorphic to $\mathbb{R}^n$.
(2) ω is supported in U that is diffeomorphic to $\mathbb{R}^n_+$.
(3) General case.

Case (1): Since $\omega = 0$ on ∂M, $\int_{\partial M} \omega = 0$. To calculate $\int_M d\omega$, we denote

$$\omega = \sum_i (-1)^{i-1} f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

where f_i's are compactly supported function. Then

$$d\omega = \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n.$$

So by definition,

$$\int_M d\omega = \int_{\mathbb{R}^n} \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^n = \sum_i \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n = 0.$$

Case (2): We have the same formula to calculate $\int_M d\omega$, except for the last term (i.e. $i = n$ term), where instead of 0 we will get

$$\int_{\mathbb{R}^{n-1}} \left(\int_0^{\infty} \frac{\partial f_n}{\partial x^n} dx^n \right) dx^1 \cdots dx^{n-1} = - \int_{\mathbb{R}^{n-1}} f_n(x^1, \cdots, x^{n-1}, 0) dx^1 \cdots dx^n.$$

On the other hand, since $x^n = 0$ on ∂M, we see

$$\omega|_{\partial M} = (-1)^{n-1} f_n(x^1, \cdots, x^{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1} = -f_n(x^1, \cdots, x^{n-1}, 0) \cdot (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}.$$

So

$$\int_{\partial M} \omega = \int_{\mathbb{R}^{n-1}} (-f_n(x^1, \cdots, x^{n-1}, 0)) dx^1 \cdots dx^n = - \int_{\mathbb{R}^{n-1}} f_n(x^1, \cdots, x^{n-1}, 0) dx^1 \cdots dx^n.$$

Case (3): In general we take a partition of unity as usual. Then

$$\int_{\partial M} \omega = \int_{\partial M} \sum_\alpha \rho_\alpha \omega = \sum_\alpha \int_M d(\rho_\alpha \omega) = \sum_\alpha \int_M d\rho_\alpha \wedge \omega + \int_M d\omega.$$

Now the conclusion follows from the fact

$$\sum_\alpha \int_M d\rho_\alpha \wedge \omega = \int_M d(\sum \rho_\alpha) \wedge \omega = 0.$$

□