LECTURE 19: THE DE RHAM COHOMOLOGY

1. The De Rham cohomology

Let M be a smooth manifold. As we have seen, $d : \Omega^k(M) \to \Omega^{k+1}(\mathrm{M})$ is a linear map so that for any k and any $\omega \in \Omega^k(M)$, $d^2\omega = d(d\omega) = 0$. So we have a "de Rham complex"

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

Definition 1.1. (1) A k-form ω is *closed* if $d\omega = 0$.
(2) A k-form ω is *exact* if there exists a $(k-1)$-form η such that $\omega = d\eta$.

We will denote the set of closed k-forms by $Z^k(M)$, and the set of exact k-forms by $B^k(M)$:

$$Z^k(M) = \mathrm{Ker}(d : \Omega^k(M) \to \Omega^{k+1}(M)),$$
$$B^k(M) = \mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M)).$$

Obviously both $Z^k(M)$ and $B^k(M)$ are vector subspaces of $\Omega^k(M)$. Moreover, since $d^2 = 0$, we have $B^k(M) \subset Z^k(M)$ as an additive subgroup.

Definition 1.2. The quotient group

$$H^k_{dR}(M) := Z^k(M)/B^k(M)$$

is called the k^{th} *de Rham cohomology group* of M.

Given any $\omega \in Z^k(M)$, we will denote by $[\omega]$ the corresponding cohomology class.

Note that for $k > n$, $B^k(M) = Z^k(M) = \{0\}$, so $H^k_{dR}(M) = \{0\}$.

Definition 1.3. In the case $\dim H^k_{dR}(M) < \infty$, we will call the number

$$b_k(M) = \dim H^k_{dR}(M)$$

the k^{th} *Betti number* of M, and the number

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b_k(M)$$

the *Euler characteristic* of M.

Remark. We will see that if M is compact, or homotopic equivalent to a compact manifold, then all its de Rham cohomology groups are finitely dimensional. On the other hand, it is not hard to construct a manifold with infinitely dimensional cohomology group, e.g. $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

Note that $B^0(M) = \{0\}$, so

$$H^0_{dR}(M) = Z^0(M) = \{f \in C^\infty(M) \mid df = 0\}.$$

As a consequence, we get

Corollary 1.4. *If M is connected, then $H^0_{dR}(M) \simeq \mathbb{R}$. More generally, if M has m connected components, then $H^0_{dR}(M) \simeq \mathbb{R}^m$, and thus $b_0(M) = m$.*

Example. For $M = \mathbb{R}$, we have $H^0_{dR}(\mathbb{R}) \simeq \mathbb{R}$ and $H^k_{dR}(\mathbb{R}) = \{0\}$ for $k \geq 2$.

To calculate $H^1_{dR}(\mathbb{R})$, we just notice that any 1-form $g(t)dt$ on $\mathbb{R}$ is exact, since

$$\omega = g(t)dt \Longleftrightarrow \omega = dG, \text{where } G(t) = \int_0^t g(\tau)d\tau.$$

It follows that $Z^1(\mathbb{R}) = \Omega^1(\mathbb{R}) = B^1(\mathbb{R})$. So $H^1_{dR}(\mathbb{R}) = \{0\}$.

Example. Consider $M = S^1$. As we have seen, $H^0_{dR}(S^1) \simeq \mathbb{R}$ and $H^k_{dR}(S^1) = 0$ for $k \geq 2$.

To calculate $H^1_{dR}(S^1)$, we argue as in the previous example. Note that on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, the "angle" variable θ is not a globally defined smooth function on S^1, but the translation invariance of d on $\mathbb{R}$ implies that the differential form $d\theta$ is a globally defined 1-form on S^1. So

$$Z^1(S^1) = \Omega^1(S^1) = \{fd\theta \mid f \in C^\infty(S^1)\} \simeq \{f \in C^\infty(\mathbb{R}) \mid f \text{ is periodic with period } 2\pi\}.$$

On the other hand, by the fundamental theorem of calculus,

$$\begin{aligned} \omega \text{ is exact} &\Longleftrightarrow \omega = df, \text{ where } f \text{ is periodic with period } 2\pi \\ &\Longleftrightarrow \omega = g(\theta)d\theta, \text{ where } g \text{ is periodic with period } 2\pi \text{ and } \int_0^{2\pi} g(\theta)d\theta = 0. \end{aligned}$$

So we conclude

$$H^1_{dR}(S^1) \simeq \frac{\{f \in C^\infty(\mathbb{R}) \mid f \text{ is periodic with period } 2\pi\}}{\{g \in C^\infty(\mathbb{R}) \mid g \text{ is periodic with period } 2\pi, \text{ and } \int_0^{2\pi} g(\theta)d\theta = 0\}}.$$

This implies that $H^1_{dR}(S^1) \simeq \mathbb{R}$, since the linear map

$$\varphi : H^1_{dR}(S^1) \to \mathbb{R}, \qquad [f] \mapsto \int_0^{2\pi} f(\theta)d\theta.$$

is an linear isomorphism:

- φ is well-defined: $[f_1] = [f] \Longrightarrow f_1 - f \in B^1(S^1) \Longrightarrow \int_0^{2\pi} f_1(\theta)d\theta = \int_0^{2\pi} f(\theta)d\theta$.
- φ is injective: $[f_1] \neq [f] \Longrightarrow f_1 - f \notin B^1(S^1) \Longrightarrow \int_0^{2\pi} f_1(\theta)d\theta \neq \int_0^{2\pi} f(\theta)d\theta$.
- φ is surjective: for any $c \in \mathbb{R}$, $f(\theta) := c \in Z^1(S^1) \Longrightarrow \varphi([f]) = \int_0^{2\pi} f(\theta)d\theta = 2\pi c$.

One can extend the wedge product and pull-back operations on forms to cohomology groups. First let $\omega \in Z^k(M)$ and $\eta \in Z^l(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = 0,$$

i.e. $\omega \wedge \eta \in Z^{k+l}(M)$. Moreover, for any $\xi_1 \in \Omega^{k-1}(M)$ and $\xi_2 \in \Omega^{l-1}(M)$,

$$(\omega + d\xi_1) \wedge (\eta + d\xi_2) = \omega \wedge \eta + d\left[(-1)^k \omega \wedge \xi_2 + (-1)^{k-1}\xi_1 \wedge \eta + (-1)^{k-1}\xi_1 \wedge d\xi_2\right].$$

In other words, $[\omega \wedge \eta]$ is independent of the choice of ω and η in $[\omega]$ and $[\eta]$. So we can define

Definition 1.5. The *cup product* between $[\omega] \in H^k_{dR}(M)$ and $[\eta] \in H^l_{dR}(M)$ is

$$[\omega] \cup [\eta] := [\omega \wedge \eta].$$

Similarly suppose $\varphi : M \to N$ is smooth. Then the fact $d\varphi^* = \varphi^* d$ implies

$$\varphi^*(Z^k(N)) \subset Z^k(M) \quad \text{and} \quad \varphi^*(B^k(N)) \subset B^k(M).$$

It follows that $\varphi^* : \Omega^k(N) \to \Omega^k(M)$ descends to a pull-back $\varphi^* : H^k_{dR}(N) \to H^k_{dR}(M)$:

$$\varphi^*([\omega]) := [\varphi^*\omega].$$

Obviously φ^* is a group homomorphism. It is easy to check

- $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
- $\mathrm{Id}^* = \mathrm{Id}$.

As an immediate consequence, we see that the de Rham cohomology is a smooth invariant:

Theorem 1.6 (Diffeomorphism Invariance)**.** *If $\varphi : M \to N$ is a diffeomorphism, then $\varphi^* : H^k_{dR}(N) \to H^k_{dR}(M)$ is a linear isomorphism for all k. In particular, $b_k(N) = b_k(M)$ for all k, and $\chi(N) = \chi(M)$.*

Remark. For any smooth map $\varphi : M \to N$, The cup product makes $H^*_{dR}(M) = \oplus_{k=0}^n H^k_{dR}(M)$ a graded ring, and the induced map φ^* is in fact a ring homomorphism from $H^*_{dR}(N)$ to $H^*_{dR}(M)$ since $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$. So if φ is a diffeomorphism, then $\varphi^* : H^*_{dR}(N) \to H^*_{dR}(M)$ is a ring isomorphism.

2. Homotopic Invariance

Let X is a smooth vector field on M, then X generate a local flow, i.e. a family of diffeomorphisms $\phi_t : M \to M$ for $|t|$ small so that $\phi_t \circ \phi_s = \phi_{t+s}$. In PSet 3 we studied the Lie derivatives of functions and vector fields with respect to X, and proved

$$\mathcal{L}_X f := \left.\frac{d}{dt}\right|_{t=0} \phi_t^* f = Xf,$$

$$\mathcal{L}_X Y := \left.\frac{d}{dt}\right|_{t=0} (\phi_{-t})_* Y = [X, Y].$$

Now we can define the Lie derivative of differential forms. (Note: functions are 0-forms.)

Definition 2.1. The *Lie derivative* of $\omega \in \Omega^k(M)$ with respect to vector field X is

$$\mathcal{L}_X \omega = \left.\frac{d}{dt}\right|_{t=0} \phi_t^* \omega.$$

By using the facts $d\varphi^* = \varphi^* d$ and $\varphi^*(\omega \wedge \eta) = \phi^*\omega \wedge \phi^*\eta$ we immediately get

Lemma 2.2. (1) $d\mathcal{L}_X\omega = \mathcal{L}_X d\omega$.

(2) $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X\omega \wedge \eta + \omega \wedge \mathcal{L}_X\eta$.

Recall that by definition,

$$\iota_X df = df(X) = X(f) = \mathcal{L}_X f.$$

This can be generated to the following extremely useful formula:

Theorem 2.3 (Cartan's magic formula)**.** *For any $\omega \in \Omega^k(M)$ and any $X \in \Gamma^\infty(TM)$,*

$$\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega.$$

Proof. We proceed by induction. We have seen that the formula holds for $k = 0$. Now suppose the formula holds for all $(k-1)$-forms. Let ω be a k-form. By linearity, we may assume that locally $\omega = f dx^1 \wedge \cdots \wedge dx^k = dx^1 \wedge \omega_1$, where $\omega_1 = f dx^2 \wedge \cdots \wedge dx^k$. Then

$$\begin{aligned} d\iota_X(dx^1 \wedge \omega_1) + \iota_X d(dx^1 \wedge \omega_1) =& d((\iota_X dx^1)\omega_1 - dx^1 \wedge \iota_X\omega_1)) - \iota_X(dx^1 \wedge d\omega_1) \\ =& d(\iota_X(dx^1)) \wedge \omega_1 + (\iota_X dx^1)d\omega_1 + dx^1 \wedge d(\iota_X\omega_1) \\ & - (\iota_X dx^1)d\omega_1 + dx^1 \wedge \iota_X(d\omega_1) \\ =& d(\mathcal{L}_X x^1) \wedge \omega_1 + dx^1 \wedge \mathcal{L}_X\omega_1 \\ =& (\mathcal{L}_X dx^1) \wedge \omega_1 + dx^1 \wedge \mathcal{L}_X\omega_1 \\ =& \mathcal{L}_X\omega. \end{aligned}$$

□

As a consequence, we get

Corollary 2.4. *Let X be a complete vector field on M, and ϕ_t the flow generated by X. Then there is a linear operator $Q : \Omega^k(M) \to \Omega^{k-1}(M)$ so that for any $\omega \in \Omega^k(M)$,*

$$\phi_1^*\omega - \omega = dQ(\omega) + Q(d\omega).$$

Proof. If we set $Q_t(\omega) = \iota_X(\phi_t^*\omega)$, then

$$\begin{aligned} \frac{d}{dt}\phi_t^*\omega &= \left.\frac{d}{ds}\right|_{s=0} \phi_{t+s}^*\omega = \left.\frac{d}{ds}\right|_{s=0} \phi_s^*\phi_t^*\omega \\ &= \mathcal{L}_X(\phi_t^*\omega) = d\iota_X(\phi_t^*\omega) + \iota_X d(\phi_t^*\omega) \\ &= d(Q_t\omega) + \iota_X\phi_t^*(d\omega) = d(Q_t\omega) + Q_t(d\omega). \end{aligned}$$

So if we denote $Q(\omega) = \int_0^1 Q_t(\omega)dt$, then

$$\phi_1^*\omega - \omega = \int_0^1 \left(\frac{d}{dt}\phi_t^*\omega\right) dt = dQ(\omega) + Q(d\omega).$$

□

Now we are ready to prove the homotopy invariance of the de Rham cohomology group. Recall

Definition 2.5. Two smooth maps $f : M \to N$ and $g : M \to N$ are said to be *homotopic* if there is a smooth map $F : M \times \mathbb{R} \to N$ so that for all $p \in M$, $F(p, 0) = f(p)$ and $F(p, 1) = g(p)$.

Theorem 2.6. *Let $f, g : M \to N$ be homotopic, then $f^* = g^* : H^k_{dR}(N) \to H^k_{dR}(M)$.*

Proof. Let $W = M \times \mathbb{R}$, then $X = \frac{\partial}{\partial t}$ is a complete vector field on W whose flow is

$$\phi_t(p, a) = (p, a + t).$$

Then according to corollary 2.4, there is a linear operator $Q : \Omega^k(W) \to \Omega^{k-1}(W)$ so that

$$\phi_1^*\omega - \omega = dQ(\omega) + Q(d\omega).$$

We let $\iota : M \to W$ be the map $\iota(p) = (p, 0)$, then $f = F \circ \iota$ and $g = F \circ \phi_1 \circ \iota$, where $F : W \to N$ is the homotopy between f and g. It follows that for any $\omega \in \Omega^k(N)$,

$$g^*\omega - f^*\omega = \iota^*\phi_1^*F^*\omega - \iota^*F^*\omega = \iota^*(dQ + Qd)F^*\omega = (d\iota^*QF^* + \iota^*QF^*d)\omega.$$

So if we denote $\widetilde{Q} = \iota^*QF^*$, then for any $\omega \in Z^k(N)$,

$$g^*\omega - f^*\omega = d\widetilde{Q}\omega - \widetilde{Q}d\omega = d\widetilde{Q}(\omega),$$

and thus $g^*([\omega]) = [g^*(\omega)] = [f^*(\omega)] = f^*([\omega])$. □

Definition 2.7. Two smooth manifolds M and N are said to be *homotopy equivalent* if there exists smooth maps $\varphi : M \to N$ and $\psi : N \to M$ so that $\varphi \circ \psi$ is homotopic to Id_N and $\psi \circ \varphi$ is homotopic to Id_M.

Example. S^{n-1} is homotopic equivalent to $\mathbb{R}^n \setminus \{0\}$ via maps $\iota : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ and

$$\psi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}, \quad x \mapsto \frac{x}{|x|}.$$

Clearly $\psi \circ \iota = \mathrm{Id}_{S^{n-1}}$. The map $\iota \circ \psi(x) = \frac{x}{|x|}$ is homotopic to $\mathrm{Id}_{\mathbb{R}^n \setminus \{0\}}$ via

$$\Phi(x, t) = tx + (1 - t)\frac{x}{|x|}.$$

Example. A *star-shaped region* in $\mathbb{R}^n$ is a region $U \subset \mathbb{R}^n$ satisfying the property that there exists some $x_0 \in U$ so that for all $x \in U$ and all $0 \le t \le 1$, one has $tx_0 + (1 - t)x \in U$. Obviously a star-shaped region is homotopic equivalent to a single point set $\{x_0\}$. (What are the maps that give the equivalence?) Any region that is homotopic to a single point set is called *contractible.*

Theorem 2.8 (Homotopy Invariance)**.** *If M and N are homotopic equivalent, then for all k, $H^k_{dR}(M) \simeq H^k_{dR}(N)$.*

Proof. Applying theorem 2.6, we get

$$\varphi^* \circ \psi^* = \mathrm{Id} : H^k_{dR}(M) \to H^k_{dR}(M)$$
$$\psi^* \circ \varphi^* = \mathrm{Id} : H^k_{dR}(N) \to H^k_{dR}(N).$$

It follows that φ^* and ψ^* are linear isomorphisms [and in fact ring isomorphisms]. □

As consequences, we immediately get

Corollary 2.9 (Poincare's lemma)**.** *If U is a star-shaped region in $\mathbb{R}^n$, then for any $k \geq 1$, $H^k_{dR}(U) = 0$. In particular, $H^k_{dR}(\mathbb{R}^n) = 0$ for all $k \geq 1$.*

Corollary 2.10 (Topological Invariance)**.** *If M is homeomorphic to N, then $H^k_{dR}(M) \simeq H^k_{dR}(N)$ for all k.*

Remark. Although in defining $H^k_{dR}(M)$, we need to use the smooth structure on M (to define d, $\Omega^k(M)$ etc), the last corollary tells us that $H^k_{dR}(M)$ only depends on the topology of M, and is independent of the smooth structure! In fact, for any topological space X one can define a *singular cohomology groups* $H^k_{sing}(X, \mathbb{R})$ of X which depends only on the topology of X. The famous theorem of de Rham claims

Theorem 2.11 (The de Rham theorem)**.** $H^k_{dR}(M) = H^k_{sing}(M, \mathbb{R})$ *for all k.*