LECTURE 19: THE DE RHAM COHOMOLOGY

1. The De Rham cohomology

Let $M$ be a smooth manifold. As we have seen, $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is a linear map so that for any $k$ and any $\omega \in \Omega^k(M)$, $d^2 \omega = d(d\omega) = 0$. So we have a “de Rham complex”

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

**Definition 1.1.** (1) A $k$-form $\omega$ is *closed* if $d\omega = 0$.

(2) A $k$-form $\omega$ is *exact* if there exists a $(k-1)$-form $\eta$ such that $\omega = d\eta$.

We will denote the set of closed $k$-forms by $Z^k(M)$, and the set of exact $k$-forms by $B^k(M)$:

$$Z^k(M) = \text{Ker}(d : \Omega^k(M) \to \Omega^{k+1}(M)),$$

$$B^k(M) = \text{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M)).$$

Obviously both $Z^k(M)$ and $B^k(M)$ are vector subspaces of $\Omega^k(M)$. Moreover, since $d^2 = 0$, we have $B^k(M) \subset Z^k(M)$ as an additive subgroup.

**Definition 1.2.** The quotient group

$$H^k_{dR}(M) := Z^k(M)/B^k(M)$$

is called the $k^{th}$ de Rham cohomology group of $M$.

Given any $\omega \in Z^k(M)$, we will denote by $[\omega]$ the corresponding cohomology class.

Note that for $k > n$, $B^k(M) = Z^k(M) = \{0\}$, so $H^k_{dR}(M) = \{0\}$.

**Definition 1.3.** In the case $\dim H^k_{dR}(M) < \infty$, we will call the number

$$b_k(M) = \dim H^k_{dR}(M)$$

the $k^{th}$ Betti number of $M$, and the number

$$\chi(M) = \sum_{k=0}^n (-1)^k b_k(M)$$

the Euler characteristic of $M$.

**Remark.** We will see that if $M$ is compact, or homotopic equivalent to a compact manifold, then all its de Rham cohomology groups are finitely dimensional. On the other hand, it is not hard to construct a manifold with infinitely dimensional cohomology group, e.g. $\mathbb{R}^2 \setminus \mathbb{Z}^2$. 

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Note that $B^0(M) = \{0\}$, so
\[ H^0_{dr}(M) = Z^0(M) = \{ f \in C^\infty(M) \mid df = 0 \}. \]

As a consequence, we get

**Corollary 1.4.** If $M$ is connected, then $H^0_{dr}(M) \simeq \mathbb{R}$. More generally, if $M$ has $m$ connected components, then $H^0_{dr}(M) \simeq \mathbb{R}^m$, and thus $b_0(M) = m$.

**Example.** For $M = \mathbb{R}$, we have $H^0_{dr}(\mathbb{R}) \simeq \mathbb{R}$ and $H^k_{dr}(\mathbb{R}) = \{0\}$ for $k \geq 2$.

To calculate $H^1_{dr}(\mathbb{R})$, we just notice that any 1-form $g(t)dt$ on $\mathbb{R}$ is exact, since
\[ \omega = g(t)dt \iff \omega = dG, \text{ where } G(t) = \int_0^t g(\tau)d\tau. \]

It follows that $Z^1(\mathbb{R}) = \Omega^1(\mathbb{R}) = B^1(\mathbb{R})$. So $H^1_{dr}(\mathbb{R}) = \{0\}$.

**Example.** Consider $M = S^1$. As we have seen, $H^0_{dr}(S^1) \simeq \mathbb{R}$ and $H^k_{dr}(S^1) = 0$ for $k \geq 2$.

To calculate $H^1_{dr}(S^1)$, we argue as in the previous example. Note that on $S^1 = \mathbb{R}/2\pi \mathbb{Z}$, the “angle” variable $\theta$ is not a globally defined smooth function on $S^1$, but the translation invariance of $d$ on $\mathbb{R}$ implies that the differential form $d\theta$ is a globally defined 1-form on $S^1$. So
\[ Z^1(S^1) = \Omega^1(S^1) = \{ f d\theta \mid f \in C^\infty(S^1) \} \simeq \{ f \in C^\infty(\mathbb{R}) \mid f \text{ is periodic with period } 2\pi \}. \]

On the other hand, by the fundamental theorem of calculus,
\[ \omega \text{ is exact } \iff \omega = df, \text{ where } f \text{ is periodic with period } 2\pi \]
\[ \iff \omega = g(\theta)d\theta, \text{ where } g \text{ is periodic with period } 2\pi \text{ and } \int_0^{2\pi} g(\theta)d\theta = 0. \]

So we conclude
\[ H^1_{dr}(S^1) \simeq \frac{\{ f \in C^\infty(\mathbb{R}) \mid f \text{ is periodic with period } 2\pi \}}{\{ g \in C^\infty(\mathbb{R}) \mid g \text{ is periodic with period } 2\pi, \text{ and } \int_0^{2\pi} g(\theta)d\theta = 0 \}}. \]

This implies that $H^1_{dr}(S^1) \simeq \mathbb{R}$, since the linear map
\[ \varphi : H^1_{dr}(S^1) \rightarrow \mathbb{R}, \quad [f] \mapsto \int_0^{2\pi} f(\theta)d\theta. \]

is an linear isomorphism:

- $\varphi$ is well-defined: $[f_1] = [f] \implies f_1 - f \in B^1(S^1) \implies \int_0^{2\pi} f_1(\theta)d\theta = \int_0^{2\pi} f(\theta)d\theta$.
- $\varphi$ is injective: $[f_1] \neq [f] \implies f_1 - f \notin B^1(S^1) \implies \int_0^{2\pi} f_1(\theta)d\theta \neq \int_0^{2\pi} f(\theta)d\theta$.
- $\varphi$ is surjective: for any $c \in \mathbb{R}$, $f(\theta) := c \in Z^1(S^1) \implies \varphi([f]) = \int_0^{2\pi} f(\theta)d\theta = 2\pi c$.

One can extend the wedge product and pull-back operations on forms to cohomology groups. First let $\omega \in Z^k(M)$ and $\eta \in Z^l(M)$, then
\[ d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = 0, \]
i.e. $\omega \wedge \eta \in Z^{k+l}(M)$. Moreover, for any $\xi_1 \in \Omega^{k-1}(M)$ and $\xi_2 \in \Omega^{l-1}(M)$,

$$(\omega + d\xi_1) \wedge (\eta + d\xi_2) = \omega \wedge \eta + d[(\omega \wedge \xi_2 + (-1)^k\omega \wedge \xi_1 + (\omega \wedge \xi_1) - (-1)^{k-1}\xi_1 \wedge d\xi_2].$$

In other words, $[\omega \wedge \eta]$ is independent of the choice of $\omega$ and $\eta$ in $[\omega]$ and $[\eta]$. So we can define

**Definition 1.5.** The cup product between $[\omega] \in H^k_{dR}(M)$ and $[\eta] \in H^l_{dR}(M)$ is

$$[\omega] \cup [\eta] := [\omega \wedge \eta].$$

Similarly suppose $\varphi : M \to N$ is smooth. Then the fact $d\varphi^* = \varphi^*d$ implies

$$\varphi^*(Z^k(N)) \subset Z^k(M) \quad \text{and} \quad \varphi^*(B^k(N)) \subset B^k(M).$$

It follows that $\varphi^* : \Omega^k(N) \to \Omega^k(M)$ descends to a pull-back $\varphi^* : H^k_{dR}(N) \to H^k_{dR}(M)$:

$$\varphi^*([\omega]) := [\varphi^*\omega].$$

Obviously $\varphi^*$ is a group homomorphism. It is easy to check

- $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
- $\text{Id}^* = \text{Id}$.

As an immediate consequence, we see that the de Rham cohomology is a smooth invariant:

**Theorem 1.6 (Diffeomorphism Invariance).** If $\varphi : M \to N$ is a diffeomorphism, then $\varphi^* : H^k_{dR}(N) \to H^k_{dR}(M)$ is a linear isomorphism for all $k$. In particular, $b_k(N) = b_k(M)$ for all $k$, and $\chi(N) = \chi(M)$.

**Remark.** For any smooth map $\varphi : M \to N$, The cup product makes $H^*_{dR}(M) = \bigoplus_{k=0}^n H^k_{dR}(M)$ a graded ring, and the induced map $\varphi^*$ is in fact a ring homomorphism from $H^*_{dR}(N)$ to $H^*_{dR}(M)$ since $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$. So if $\varphi$ is a diffeomorphism, then $\varphi^* : H^*_{dR}(N) \to H^*_{dR}(M)$ is a ring isomorphism.

### 2. Homotopic Invariance

Let $X$ be a smooth vector field on $M$, then $X$ generate a local flow, i.e. a family of diffeomorphisms $\phi_t : M \to M$ for $|t|$ small so that $\phi_t \circ \phi_s = \phi_{t+s}$. In PSet 3 we studied the Lie derivatives of functions and vector fields with respect to $X$, and proved

$$\mathcal{L}_X f := \left. \frac{d}{dt} \right|_{t=0} \phi_t^* f = Xf,$$

$$\mathcal{L}_X Y := \left. \frac{d}{dt} \right|_{t=0} (\phi_{-t})_* Y = [X,Y].$$

Now we can define the Lie derivative of differential forms. (Note: functions are 0-forms.)

**Definition 2.1.** The Lie derivative of $\omega \in \Omega^k(M)$ with respect to vector field $X$ is

$$\mathcal{L}_X \omega = \left. \frac{d}{dt} \right|_{t=0} \phi_t^* \omega.$$

By using the facts $d\varphi^* = \varphi^*d$ and $\varphi^*(\omega \wedge \eta) = \phi^* \omega \wedge \phi^* \eta$ we immediately get
**Lemma 2.2.** (1) \(d\mathcal{L}_X \omega = \mathcal{L}_X d\omega\).
(2) \(\mathcal{L}_X (\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta\).

Recall that by definition,
\[\iota_X df = df(\mathcal{X}) = \mathcal{X}(f) = \mathcal{L}_X f.\]
This can be generated to the following extremely useful formula:

**Theorem 2.3** (Cartan’s magic formula). For any \(\omega \in \Omega^k(M)\) and any \(X \in \Gamma(\infty(TM))\),
\[\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega.\]

*Proof.* We proceed by induction. We have seen that the formula holds for \(k = 0\). Now suppose the formula holds for all \((k - 1)-\)forms. Let \(\omega\) be a \(k\)-form. By linearity, we may assume that locally \(\omega = f dx^1 \wedge \cdots \wedge dx^k = dx^1 \wedge \omega_1\), where \(\omega_1 = f dx^2 \wedge \cdots \wedge dx^k\). Then
\[
d\iota_X (dx^1 \wedge \omega_1) + \iota_X d(dx^1 \wedge \omega_1) = d((\iota_X dx^1)\omega_1 - dx^1 \wedge \iota_X \omega_1)) - \iota_X (dx^1 \wedge d\omega_1)
= d((\iota_X dx^1)) \wedge \omega_1 + (\iota_X dx^1)d\omega_1 + dx^1 \wedge d(\iota_X \omega_1)
- (\iota_X dx^1)d\omega_1 + dx^1 \wedge \iota_X (d\omega_1)
= d(\mathcal{L}_X x^1) \wedge \omega_1 + dx^1 \wedge \mathcal{L}_X \omega_1
= (\mathcal{L}_X dx^1) \wedge \omega_1 + dx^1 \wedge \mathcal{L}_X \omega_1
= \mathcal{L}_X \omega.
\]

\[\square\]

As a consequence, we get

**Corollary 2.4.** Let \(X\) be a complete vector field on \(M\), and \(\phi_t\) the flow generated by \(X\). Then there is a linear operator \(Q : \Omega^k(M) \to \Omega^{k-1}(M)\) so that for any \(\omega \in \Omega^k(M)\),
\[\phi_t^* \omega - \omega = dQ(\omega) + Q(d\omega).\]

*Proof.* If we set \(Q_t(\omega) = \iota_X (\phi_t^* \omega)\), then
\[
\frac{d}{dt} \phi_t^* \omega = \frac{d}{ds}\bigg|_{s=0} \phi_{t+s}^* \omega = \frac{d}{ds}\bigg|_{s=0} \phi_s^* \phi_t^* \omega
= \mathcal{L}_X (\phi_t^* \omega) = d\iota_X (\phi_t^* \omega) + \iota_X d(\phi_t^* \omega)
= d(Q_t(\omega) + \iota_X \phi_t^* (d\omega)) = d(Q_t(\omega) + Q_t(d\omega)).
\]

So if we denote \(Q(\omega) = \int_0^1 Q_t(\omega) dt\), then
\[
\phi_t^* \omega - \omega = \int_0^1 \left(\frac{d}{dt} \phi_t^* \omega\right) dt = dQ(\omega) + Q(d\omega).
\]

\[\square\]

Now we are ready to prove the homotopy invariance of the de Rham cohomology group. Recall
Definition 2.5. Two smooth maps \( f : M \to N \) and \( g : M \to N \) are said to be homotopic if there is a smooth map \( F : M \times \mathbb{R} \to N \) so that for all \( p \in M \), \( F(p, 0) = f(p) \) and \( F(p, 1) = g(p) \).

Theorem 2.6. Let \( f, g : M \to N \) be homotopic, then \( f^* = g^* : H^k_{\text{dR}}(N) \to H^k_{\text{dR}}(M) \).

Proof. Let \( W = M \times \mathbb{R} \), then \( X = \frac{\partial}{\partial t} \) is a complete vector field on \( W \) whose flow is \( \phi_t(p, a) = (p, a + t) \).

Then according to corollary 2.4, there is a linear operator \( Q : \Omega^k(W) \to \Omega^{k-1}(W) \) so that \( \phi_t^* \omega - \omega = dQ(\omega) + Q(d\omega) \).

We let \( \iota : M \to W \) be the map \( \iota(p) = (p, 0) \), then \( f = F \circ \iota \) and \( g = F \circ \phi_1 \circ \iota \), where \( F : W \to N \) is the homotopy between \( f \) and \( g \). It follows that for any \( \omega \in \Omega^k(N) \),

\[
g^* \omega - f^* \omega = \iota^* \phi_1^* F^* \omega - \iota^* F^* \omega = \iota^*(dQ + Qd)F^* \omega = (d\iota^* QF^* + \iota^* QF^* d) \omega.
\]

So if we denote \( Q = \iota^* QF^* \), then for any \( \omega \in Z^k(N) \),

\[
g^* \omega - f^* \omega = dQ \omega - \tilde{Q} d \omega = dQ (\omega),
\]

and thus \( g^*([\omega]) = [g^*(\omega)] = [f^*(\omega)] = f^*([\omega]) \).

Definition 2.7. Two smooth manifolds \( M \) and \( N \) are said to be homotopy equivalent if there exists smooth maps \( \varphi : M \to N \) and \( \psi : N \to M \) so that \( \varphi \circ \psi \) is homotopic to \( \text{Id}_N \) and \( \psi \circ \varphi \) is homotopic to \( \text{Id}_M \).

Example. \( S^{n-1} \) is homotopy equivalent to \( \mathbb{R}^n \setminus \{0\} \) via maps \( \iota : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\} \) and \( \psi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}, \quad x \mapsto \frac{x}{|x|} \).

Clearly \( \psi \circ \iota = \text{Id}_{S^{n-1}} \). The map \( \iota \circ \psi(x) = \frac{x}{|x|} \) is homotopic to \( \text{Id}_{\mathbb{R}^n \setminus \{0\}} \) via \( \Phi(x,t) = tx + (1-t)\frac{x}{|x|} \).

Example. A star-shaped region in \( \mathbb{R}^n \) is a region \( U \subset \mathbb{R}^n \) satisfying the property that there exists some \( x_0 \in U \) so that for all \( x \in U \) and all \( 0 \leq t \leq 1 \), one has \( tx_0 + (1-t)x \in U \). Obviously a star-shaped region is homotopy equivalent to a single point set \( \{x_0\} \). (What are the maps that give the equivalence?) Any region that is homotopic to a single point set is called contractible.

Theorem 2.8 (Homotopy Invariance). If \( M \) and \( N \) are homotopy equivalent, then for all \( k \), \( H^k_{\text{dR}}(M) \simeq H^k_{\text{dR}}(N) \).

Proof. Applying theorem 2.6, we get

\[
\varphi^* \circ \psi^* = \text{Id} : H^k_{\text{dR}}(M) \to H^k_{\text{dR}}(M)
\]

\[
\psi^* \circ \varphi^* = \text{Id} : H^k_{\text{dR}}(N) \to H^k_{\text{dR}}(N).
\]

It follows that \( \varphi^* \) and \( \psi^* \) are linear isomorphisms [and in fact ring isomorphisms].

As consequences, we immediately get
Corollary 2.9 (Poincare’s lemma). If $U$ is a star-shaped region in $\mathbb{R}^n$, then for any $k \geq 1$, $H^k_{dR}(U) = 0$. In particular, $H^k_{dR}(\mathbb{R}^n) = 0$ for all $k \geq 1$.

Corollary 2.10 (Topological Invariance). If $M$ is homeomorphic to $N$, then $H^k_{dR}(M) \cong H^k_{dR}(N)$ for all $k$.

Remark. Although in defining $H^k_{dR}(M)$, we need to use the smooth structure on $M$ (to define $d$, $\Omega^k(M)$ etc), the last corollary tells us that $H^k_{dR}(M)$ only depends on the topology of $M$, and is independent of the smooth structure! In fact, for any topological space $X$ one can define a singular cohomology groups $H^k_{sing}(X, \mathbb{R})$ of $X$ which depends only on the topology of $X$. The famous theorem of de Rham claims

Theorem 2.11 (The de Rham theorem). $H^k_{dR}(M) = H^k_{sing}(M, \mathbb{R})$ for all $k$. 