LECTURE 22: MAPPING DEGREE, POINCARE DUALITY

1. The mapping degree and its applications

Let M, N be n-dimensional connected oriented manifolds, and $f : M \to N$ a proper map. (If M is compact, then any smooth map $f : M \to N$ is proper.) Then the pull-back map

$$f^* : \mathbb{R} = H_c^n(N) \to H_c^n(M) = \mathbb{R}$$

is a linear map, and thus is a map of the form $\lambda \mapsto c\lambda$. The constant c is called the degree of f:

Definition 1.1. The *degree* of a proper smooth map $f : M \to N$ is the number $\deg(f)$ so that

$$\int_M f^*\omega = \deg(f) \int_N \omega, \qquad \forall \omega \in \Omega_c^n(N).$$

Remark. It is not hard to see

- If $f : M \to N$ and $g : N \to P$ are both proper, then $\deg(g \circ f) = \deg(f)\deg(g)$.
- If f and g are properly homotopic, then $\deg(f) = \deg(g)$.

Proposition 1.2. *If a proper map $f : M \to N$ is not surjective, then* $\deg(f) = 0$.

Proof. Since any proper continuous map to a manifold is closed (c.f. Lee, appendix A, theorem A.57), $f(M)$ is a closed subset in N. If $q \notin f(M)$, then one can find an open neighborhood $\widetilde{U}$ of q so that $\widetilde{U} \cap f(M) = \emptyset$. We pick an n-form ω supported in $\widetilde{U}$ so that $\int_N \omega = 1$. Since by our construction, $f^*\omega = 0$, we conclude $\deg(f) = 0$. □

Recall (lecture 18): if f is an orientation-preserving diffeomorphism, then $\int_M f^*\omega = \int_N \omega$. So

Lemma 1.3. *If $f : M \to N$ is a diffeomorphism, then*

$$\deg(f) = \begin{cases} 1, & f \text{ is orientation preserving,} \\ -1, & f \text{ is orientation reversing.} \end{cases}$$

This fact implies the following remarkable property of the degree:

$\deg(f)$ is always an integer!

To see this, we may assume that f is surjective. According to Sard's theorem, almost every point $q \in N$ is a regular value of f. In what follows we fix a regular value q of f. Since $\dim M = \dim N$, f is locally a diffeomorphism at each point in $f^{-1}(q)$. It follows that $f^{-1}(q)$ is discrete. On the other hand, by properness, $f^{-1}(q)$ is compact. It follows that $f^{-1}(q)$ is a finite set. We will denote

$$f^{-1}(q) = \{p_1, \cdots, p_k\}.$$

We can take (why?) a neighborhood $\widetilde{U}$ of q and neighborhoods U_i of p_i, $1 \le i \le k$, so that

- Each U_i is the domain of an oriented chart $\{\varphi, U_i, V_i\}$ on M.
- For $i \neq j$, $U_i \cap U_j = \emptyset$.
- $\widetilde{U}$ is the domain of an oriented chart $\{\psi, \widetilde{U}, \widetilde{V}\}$ on N.
- f maps U_i diffeomorphically to $\widetilde{U}$.
- $f^{-1}(\widetilde{U}) = \cup_{i=1}^k U_i$.

We let

$$\sigma_i = \begin{cases} 1, & \text{if } f : U_i \to \widetilde{U} \text{ is orientation preserving at } p_i, \\ -1, & \text{if } f : U_i \to \widetilde{U} \text{ is orientation reversing at } p_i. \end{cases}$$

Theorem 1.4. $\deg(f) = \sum_{i=1}^k \sigma_i$.

Proof. We take $\omega \in \Omega_c^n(\widetilde{U})$ so that $\int_N \omega = 1$. Then $f^*\omega$ is supported in $f^{-1}(\widetilde{U}) = \cup_{i=1}^k U_i$, and

$$\int_M f^*\omega = \sum_{i=1}^k \int_{U_i} f^*\omega = \sum_{i=1}^k \sigma_i \int_{\widetilde{U}} \omega = \sum_{i=1}^k \sigma_i.$$

The theorem follows. □

As an application of degree theory, we can prove the following "hairy ball theorem":

Theorem 1.5. *Even dimensional spheres do not admit non-vanishing smooth vector fields.*

Proof. Let $f : S^{2n} \to S^{2n}$ be the antipodal map, then by PSet 5 problem 6(a), f is orientation-reversion and thus $\deg(f) = -1$.

On the other hand, suppose X is a non-vanishing smooth vector field on $S^{2n} \subset \mathbb{R}^{2n+1}$. By normalizing the vectors, we may assume $|X_p| = 1$ for all $p \in S^{2n}$. We will think of p and X_p as vectors in $\mathbb{R}^{2n+1}$, and consider the map

$$F(p,t) = p\cos(t\pi) + X_p \sin(t\pi).$$

Then for each $t \in \mathbb{R}$, $F(\cdot, t)$ is a map from S^{2n} to S^{2n}. So F is a homotopy between $F(\cdot, 0) = \mathrm{Id}_{S^{2n}}$ and $F(\cdot, 1) = f$, the antipodal map. It follows $\deg(f) = \deg(\mathrm{Id}_{S^{2n}}) = 1$, a contradiction. □

Another application is

Theorem 1.6. *Suppose M is an n-dimensional oriented compact manifold with smooth and connected boundary ∂M, X a connected oriented $(n-1)$-manifold, and $f : \partial M \to X$ a smooth map that extends to a smooth map $g : M \to X$. Then* $\deg(f) = 0$.

Proof. Pick $\omega \in \Omega^{n-1}(X)$ so that $\int_X \omega = 1$. Then

$$\deg(f) = \deg(f) \int_X \omega = \int_{\partial M} f^*\omega = \int_{\partial M} g^*\omega = \int_M d(g^*\omega) = \int_M g^* d\omega = 0.$$

□

Remark. According to Whitney approximation theorem, one has

- (Theorem 6.26 in Lee's book) Any continuous map is homotopic to a smooth map.

- (Theorem 6.29 in Lee's book) If a continuous map f is homotopic to smooth maps g_1 and g_2, then g_1 and g_2 are smoothly homotopic.

So one can define the degree of a continuous map to be the degree of corresponding smooth maps. All the theorems we proved above apply to continuous maps. In fact, in algebraic topology, degree is developed for continuous maps: in that case there is no smoothness at all.

Corollary 1.7 (Brouwer Fixed Point Theorem). *Every continuous map from B^n (=the unit ball in $\mathbb{R}^n$) to itself has a fixed point.*

Proof. Let $F : B^n \to B^n$ be a continuous map without fixed point. Then

$$G : B^n \to S^{n-1}, \qquad x \mapsto \frac{x - F(x)}{|x - F(x)|}$$

is the extension of $g = G|_{S^{n-1}}$. By theorem 1.6, $\deg(g) = 0$.

On the other hand, the map

$$H : S^{n-1} \times [0, 1] \to S^{n-1}, \qquad (x, t) \mapsto \frac{x - tF(x)}{|x - tF(x)|}$$

is a homotopy between the identity map and g. So $\deg(g) = \deg(\mathrm{Id}) = 1$. Contradiction. □

2. The Poincaré duality and its applications

Let M be an oriented manifold of dimension n. We have the following maps

- $\cup : H^k_{dR}(M) \times H^l_c(M) \to H^{k+l}_c(M), \quad ([\omega], [\eta]) \mapsto [\omega \wedge \eta]$.
- $\int_M : H^n_c(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$.

For any $0 \le k \le n$, consider the bilinear pairing map

$$P^k_M : H^k_{dR}(M) \times H^{n-k}_c(M) \to \mathbb{R}, \quad P^k_M([\omega], [\eta]) = \int_M \omega \wedge \eta.$$

The map P^k_M induces the following *Poincaré duality operator*

$$\mathcal{P}^k_M : H^k_{dR}(M) \to \left(H^{n-k}_c(M)\right)^*, \quad \mathcal{P}^k([\omega]) = \left\{\eta \mapsto \int_M \omega \wedge \eta\right\}.$$

For example, $\mathcal{P}^0_M$ maps the element $1 \in \mathbb{R} = H^0_{dR}(M)$ to the linear map

$$\int_M : H^n_c(M) \to \mathbb{R}, \quad \eta \mapsto \int_M \eta$$

on $H^n_c(M)$, so that one can think of $\int_M$ as an element in $(H^n_c(M))^*$.

The major theorem we would like to discuss in this section is

Theorem 2.1 (Poincaré duality). *For any oriented manifold M and any k, the Poincaré duality map $\mathcal{P}^k_M$ is a linear isomorphism from $H^k_{dR}(M)$ to $\left(H^{n-k}_c(M)\right)^*$.*

Remark. If $H^{n-k}_c(M)$ is finite dimensional, then $\left(H^{n-k}_c(M)\right)^*$ is isomorphic to $H^{n-k}_c(M)$.

Remark. Any closed submanifold $S \subset M$ of co-dimension k defines an element $\int_S$ in $\left(H_c^{n-k}(M)\right)^*$, and thus defines an element in $H_{dR}^k(M)$. For example, the element in $H_{dR}^0(M)$ corresponding to the (co-dimension zero sub-)manifold M is 1.

Example. For $M = \mathbb{R}^n$, we have $H_{dR}^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n, \\ 0, & k \neq n, \end{cases}$ and $H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k \neq 0. \end{cases}$

Example. For $M = S^n$, we have $H_c^k(S^n) = H_{dR}^k(S^n) = \begin{cases} \mathbb{R}, & k = 0, n, \\ 0, & k \neq 0, n. \end{cases}$

Example. For any compact connected oriented manifold of dimension n, we have already seen $H_{dR}^n(M) = \mathbb{R}$ and $H_c^0(M) = \mathbb{R}$.

Example. Let $M = \cup_{i \in \mathbb{N}}(i, i+1)$ be a countable union of disjoint open intervals. Then

$$H_{dR}^0(M) = \prod_{i \in \mathbb{N}} \mathbb{R} = \{(a_1, a_2, \cdots) \mid a_i \in \mathbb{R}\}$$

and

$$H_c^1(M) = \bigoplus_{i \in \mathbb{N}} \mathbb{R} = \{(a_1, a_2, \cdots) \mid a_i \in \mathbb{R}, \text{ all but finitely many } a_i's \text{ are zero } \},$$

and the Poincaré duality follows from the well-known fact in algebra: $\left(\bigoplus_{i \in \mathbb{N}} \mathbb{R}\right)^* = \prod_{i \in \mathbb{N}} \mathbb{R}$. It is also well-known that $\left(\prod_{i \in \mathbb{N}} \mathbb{R}\right)^* \neq \bigoplus_{i \in \mathbb{N}} \mathbb{R}$. So in general $\left(H_{dR}^k(M)\right)^* \neq H_c^{n-k}(M)$.

In what follows we will sketch a proof of Poincaré duality for oriented manifolds admitting a finite good cover, although the theorem holds for any oriented manifold. We need

Lemma 2.2. *The following diagram commutes:*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow H_{dR}^k(M) & \xrightarrow{\alpha} & H_{dR}^k(U) \oplus H_{dR}^k(V) & \xrightarrow{\beta} & H_{dR}^k(U \cap V) & \xrightarrow{(-1)^{k+1}\delta} & H_{dR}^{k+1}(M) \longrightarrow \cdots \\
\downarrow \mathcal{P}_M^k & & \downarrow \mathcal{P}_U^k \oplus \mathcal{P}_V^k & & \downarrow \mathcal{P}_{U \cap V}^k & & \downarrow \mathcal{P}_M^{k+1} \\
\cdots \longrightarrow H_c^{n-k}(M)^* & \xrightarrow[\alpha^*]{} & H_c^k(U)^* \oplus H_c^k(V)^* & \xrightarrow[\beta^*]{} & H_c^{n-k}(U \cap V)^* & \xrightarrow[\delta^*]{} & H_c^{n-k-1}(M)^* \rightarrow \cdots
\end{array}$$

where the bottom row is the dual of the Mayer-Vietoris sequence for compactly supported de Rham cohomology groups.

Sketch of proof of Poincaré duality for oriented manifolds M admitting a finite good cover.

We proceed by induction. The theorem holds for M admitting one good chart, in which case $M \simeq \mathbb{R}^n$, and the isomorphism follows from the two versions of Poincaré lemma that we proved.

Now suppose the theorem holds for manifolds admitting a good cover of no more than $k-1$ open sets, and suppose M admits a good cover $\{U_1, \cdots, U_k\}$. We let $U = U_1 \cup \cdots \cup U_{k-1}$ and $V = U_k$. Then U, V and $U \cap V$ all admit a good cover of no more than $k-1$ open sets. By induction hypothesis, $\mathcal{P}_U^k$, $\mathcal{P}_V^k$ and $\mathcal{P}_{U \cap V}^k$ are all isomorphisms. By the above lemma and the five lemma (see lecture 20), $\mathcal{P}_M^k$ is an isomorphism. $\square$

Since for any connected noncompact manifold, $H_c^0(M) = 0$, we get another proof of

Corollary 2.3. *For any n-dimensional connected noncompact oriented manifold M, $H_{dR}^n(M) = 0$.*

Another corollary is (the result actually holds for any manifold M)

Corollary 2.4. *For any oriented manifold M whose compact supported cohomology groups are finite dimensional, $H_c^{k+l}(M \times \mathbb{R}^l) \simeq H_c^k(M)$.*

Proof. $H_c^{k+l}(M \times \mathbb{R}^l) \simeq H_{dR}^{n+l-k-l}(M \times \mathbb{R}^l) \simeq H_{dR}^{n-k}(M) \simeq H_c^k(M)$. □

Recall that the Betti number $b_k = \dim H_{dR}^k(M)$. So

Corollary 2.5. *If M is a compact oriented manifold of dimension n, then $b_k = b_{n-k}$.*

Recall that the Euler characteristic $\chi(M) = \sum_k (-1)^k b_k$.

Corollary 2.6. *Let M be a compact oriented manifold.*

(1) *If $\dim M = 2n+1$, then $\chi(M) = 0$.*
(2) *If $\dim M = 4n+2$, then $\chi(M)$ is even.*

Proof. (1) If $\dim M = 2n+1$, we have

$$\chi(M) = \sum_{k=0}^{2n+1}(-1)^k b_k = \sum_{k=0}^{n}(-1)^k b_k + \sum_{k=n+1}^{2n+1}(-1)^k b_{2n+1-k} = \sum_{k=0}^{n}((-1)^k + (-1)^{2n+1-k})b_k = 0.$$

(2) Suppose $\dim M = 4n+2$, then the same argument yields

$$\chi(M) = \sum_{k=0}^{4n+2}(-1)^k b_k = \sum_{k=0}^{2n}((-1)^k + (-1)^{4n+2-k})b_k + b_{2n+1}.$$

Since $(-1)^k + (-1)^{4n+2-k} = \pm 2$, it remains to prove b_{2n+1} is even. This follows from the non-degeneracy of the pairing

$$P_M^{2n+1} : H_{dR}^{2n+1}(M) \times H_{dR}^{2n+1}(M) \to \mathbb{R}.$$

If fact, for any $[\omega], [\eta] \in H_{dR}^{2n+1}(M)$, we have

$$P_M^{2n+1}([\omega],[\eta]) = \int_M \omega \wedge \eta = \int_M (-1)^{2n+1}\eta \wedge \omega = -P_M^{2n+1}([\eta],[\omega]).$$

It follows that the matrix for the pairing $P_M^{2n+1} : H_{dR}^{2n+1}(M) \times H_{dR}^{2n+1}(M) \to \mathbb{R}$ is an anti-symmetric $b_{2n+1} \times b_{2n+1}$ matrix. It follows that

$$\det(P_M^{2n+1}) = \det((P_M^{2n+1})^T) = (-1)^{b_{2n+1}} \det(P_M^{2n+1}).$$

So b_{2n+1} must be an even number, otherwise $\det(P_M^{2n+1}) = 0$ and thus P_M^{2n+1} is not non-degenerate. □

Note that in particular, we proved that for a compact oriented surface $\Sigma_g = \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ of genus g, $\dim H_{dR}^1(\Sigma_g)$ is even.