GEOMETRIC STRUCTURES ON SMOOTH MANIFOLDS

We know that a *topological structure* on a set is a structure that tells us the internal organization of the set. A topological manifold is a topological space that has an extremely nice topological structure. A *smooth structure* is an extra structure on topological manifold that allows us to do analysis. For example, now we know how to differentiate a function, or more generally, how to differentiate a differential form. We know how to integrate a top form on an orientable manifold. In general, to integral a function f on an orientable manifold M, one can first fix a volume form ω and then integrate $f\omega$. This already tells us that an "integration structure" (or more precisely a *volume structure*) is an extra structure on a smooth manifold: different volume forms give us different volume structures on M.

One can think of a volume structure as a simplest *geometric structure* on M. Roughly speaking, a geometric structure on a smooth manifold M is an extra structure that gives us some geometric information of the space and usually the structure is no longer preserved by diffeomorphisms. There are many many different types of geometric structures, each type gives a branch of differential geometry. Like the volume structure, most geometric structures are given by a (local) tensor fields or related objects.

Here is an incomplete list of geometric structures that are defined using tensor fields:

- A volume structure = A volume form, i.e. a nonvanishing top form
  - One can also think of orientation as an extra geometric structure
- A Riemannian structure = A symmetric positive-definite $(0,2)$-tensor

  A $(0,2)$-tensor $g = \sum g_{ij}(x)dx^i \otimes dx^j$ is symmetric positive-definite if the matrix $(g_{ij}(x))$ is symmetric positive-definite for all x. In other words, at each x the matrix $(g_{ij}(x))$ defines an inner product structure on T_xM. Using this one can define length, angle etc. $\rightsquigarrow$ Prof. Liu's course next semester
  - A pseudo-Riemannian (or semi-Riemannian) structure = A symmetric non-degenerate $(0,2)$-tensor field
  - A sub-Riemannian structure = a symmetric $(0,2)$-tensor that is positive-definite when restricted to a given completely non-integrable distribution
- A $\boxed{\text{symplectic structure}}$ = A closed non-degenerate 2-from
  - An almost symplectic structure = A non-degenerate 2-form
- An almost complex structure = An involutive $(1,1)$-tensor

  As we mentioned in lecture 16, any $(1,1)$-tensor field can be regarded as a vector bundle homomorphism $J : TM \to TM$. It is called involutive if $J^2 = -\text{Id}$.
  - A complex structure = An integrable almost complex structure, i.e. an almost complex structure whose Nijenhuis tensor is zero.
- A Kähler structure = A compatible triple: Riemannian + symplectic + complex
- A contact structure = A (local) 1-from α on M^{2n+1} such that $\alpha \wedge (d\alpha)^n \neq 0$.
- A Poisson structure = A smooth bi-vector field $\pi \in \Gamma(\Lambda^2 TM)$ such that $[\pi, \pi] = 0$.
- A generalized almost complex structure = an almost complex structure on $TM \oplus T^*M$ that preserves some natural inner product.
- A $\boxed{\text{connection structure}}$ = *Local* connection 1-forms $\rightsquigarrow$ Curvature 2-form

1. Linear Symplectic Structure

We start with some linear algebra. Let V be an m dimensional vector space. The following theorem is the structural theorem for

Lemma 1.1. *Let Ω be a linear 2-form on V. Then there exists a basis*

$$e_1^*, \cdots, e_{2k}^*, e_{2k+1}^*, \cdots, e_m^*$$

of V^* so that

$$\Omega = e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + \cdots + e_{2k-1}^* \wedge e_{2k}^*.$$

Proof. If $\Omega = 0$, the theorem holds trivially with $k = 0$. Now suppose $\Omega \neq 0$. Let $f^1, \cdots, f^m$ be a basis of V and $f_1^*, \cdots, f_m^*$ the dual basis of V^*. Then we can express Ω as

$$\Omega = \sum_{i<j} a_{ij} f_i^* \wedge f_j^*.$$

Without loss of generality, we assume $a_{12} \neq 0$. Let

$$e_1^* = \frac{1}{a_{12}} \iota_{f^1}\Omega \quad \text{and} \quad e_2^* = \iota_{f^2}\Omega.$$

Then $e_1^*, e_2^*, f_3^*, \cdots, f_m^*$ is still a basis of V^*, and if we set $\Omega_1 = \Omega - e_1^* \wedge e_2^*$, we have

$$\begin{aligned} \iota_{f^1}\Omega_1 &= \iota_{f^1}\Omega + e_1^* \wedge \iota_{f^1} e_2^* = a_{12}e_1^* - a_{12}e_1^* = 0, \\ \iota_{f^2}\Omega_1 &= \iota_{f^2}\Omega - \iota_{f^2} e_1^* \wedge e_2^* = e_2^* - e_2^* = 0. \end{aligned}$$

So we can express Ω_1 as $\Omega_1 = \sum_{3 \leq i<j} b_{ij} f_i^* \wedge f_j^*$, so that

$$\Omega = e_1^* \wedge e_2^* + \sum_{3 \leq i<j} b_{ij} f_i^* \wedge f_j^*.$$

If $\Omega_1 = 0$, then $\Omega = e_1^* \wedge e_2^*$ and we are done. If $\Omega_1 \neq 0$, then it defines a linear 2-form on $V/\text{span}(e_1^*, e_2^*)$, and we can repeat the procedure above with Ω replaced by Ω_1 to get e_3^*, e_4^* so that

$$\Omega = e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + \sum_{5 \leq i<j} c_{ij} f_i^* \wedge f_j^*.$$

By iterating the procedure finitely many times, we will end with the desired expression. □

Remark. This is equivalent to say that there exists a basis $e_1, \cdots, e_{2k}, e_{2k+1}, \cdots, e_m$ of V so that

$$\Omega(e_{2i-1}, e_{2i}) = -\Omega(e_{2i}, e_{2i-1}) = 1, \quad 1 \leq i \leq k$$

and $\Omega(e_i, e_j) = 0$ for all other (i, j)'s.

By definition, a linear 2-form $\Omega : V \times V \to \mathbb{R}$ is an anti-symmetric bi-linear map. From the pairing Ω we can get a linear map

$$\widetilde{\Omega} : V \to V^*, \quad \widetilde{\Omega}(v) = \Omega(v, \cdot) : w \mapsto \Omega(v, w).$$

As usual we say Ω is *non-degenerate* if $\widetilde{\Omega}$ is a linear isomorphism.

Definition 1.2. A non-degenerate linear 2-form Ω on a vector space V is called a *linear symplectic structure*, and the pair (V, Ω) is called a symplectic vector space.

As a consequence of previous lemma, we get

Corollary 1.3 (Linear Darboux Theorem)**.** $\Omega \in \Lambda^2 V^*$ *is non-degenerate if and only if* $m = 2n$ *is even and there exists a basis* $e_1^*, \cdots, e_{2n}^*$ *of* V^* *so that*

$$\Omega = e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + \cdots + e_{2n-1}^* \wedge e_{2n}^*.$$

Proof. Take a basis of V^*, and thus a dual basis of V, as before. If $2k < m$, then $\widetilde{\Omega}(e_m) = 0$ in V^*, so $\widetilde{\Omega}$ is not an isomorphism. If $2k = m$, then $\widetilde{\Omega}$ maps the basis $\{e_1, e_2, \cdots, e_{2k-1}, e_{2k}\}$ of V into the basis $\{e_2^*, -e_1^*, \cdots, e_{2k}^*, -e_{2k-1}^*\}$. So $\widetilde{\Omega}$ is a linear isomorphism. $\square$

Remark. We note that if Ω is linear symplectic form, and we denote $m = 2n$, then

$$\Omega^n := \Omega \wedge \cdots \wedge \Omega = n! e_1^* \wedge e_2^* \wedge \cdots \wedge e_{2n-1}^* \wedge e_{2n}^*.$$

is a non-vanishing top form on V. It is called the *linear Liouville form.*

2. Symplectic Manifolds

Now suppose M is a smooth manifold.

Definition 2.1. A *symplectic form* on M is a 2-form $\omega \in \Lambda^2 T^* M$ on M so that

(1) ω is closed: $d\omega = 0$.
(2) ω is non-degenerate: for each $p \in M$, $\omega_p \in \Lambda^2 T_p^* M$ is non-degenerate.

The pair (M, ω) is called a *symplectic manifold.*

According to what we just proved for linear symplectic structure, we immediately get

Lemma 2.2. *If* (M, ω) *is symplectic, then* $\dim M = \dim T_p M = 2n$ *is even. Moreover,*

$$\mu_{Liouville} = \frac{1}{n!}\omega^n = \frac{1}{n!}\omega \wedge \cdots \wedge \omega$$

is a nowhere vanishing $2n$*-form, and thus a volume form on* M*.*

The volume form $\mu_{Liouville}$ is called the *Liouville volume form* on M. As a consequence,

Corollary 2.3. *Any symplectic manifold is orientable.*

Not all even orientable manifolds admit symplectic structures.

Lemma 2.4. *If (M, ω) is a compact symplectic manifold of dimension $2n$, then for any $1 \le k \le n$, the de Rham cohomology group $H^{2k}_{dR}(M) \neq 0$.*

Proof. Since ω is closed, for any $k \ge 1$, the $2k$-from ω^k must be closed, thus defines a cohomology class. We claim that $0 \neq [\omega^k] \in H^{2k}_{dR}(M)$. In fact, if $[\omega^k] = 0$, i.e. $\omega^k = d\alpha$, then

$$\omega^n = d\alpha \wedge \omega^{n-k} = d(\alpha \wedge \omega^{n-k}),$$

i.e. $[\omega^n] = 0$. So by Stokes' theorem, $\mathrm{Vol}(M) = \int_M \frac{\omega^n}{n!} = 0$, a contradiction. $\square$

Remark. Note: This result does not hold for non-compact symplectic manifold. For example, $\mathbb{R}^{2n}$.

Corollary 2.5. *There is no symplectic structure on S^m for any $m \neq 2$.*

In what follows we will give several important examples of symplectic manifolds.

Example. Let M be an oriented 2 dimensional surface, and ω is a volume form on M. Then ω is non-degenerate since it is a non-vanishing 2-form on a 2-dimensional manifold, it is also closed since $d\omega$ is a 3-form on M which has to be 0. So (M, ω) is a symplectic manifold.

Example. Let $M = \mathbb{R}^{2n}$, with coordinates $\{x^1, \cdots, x^n, y^1, \cdots, y^n\}$, then

$$\omega_0 = \sum dx^i \wedge dy^i = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$$

is a symplectic form on M. Note that in this example,

$$\frac{\omega^n}{n!} = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n$$

is the Lebesgue volume form on $\mathbb{R}^{2n}$.

Remark. We will show next time that any symplectic manifold *locally* looks like $(\mathbb{R}^{2n}, \omega_0)$.

The next example shows that although there are many restrictions for a manifold to admit a symplectic structure, we do have lots of symplectic manifolds – as many as smooth manifolds!

Example. Let X be a smooth manifold of dimension n, then its cotangent bundle $M = T^*X$ admits a natural symplectic structure, described as follows: Let $\{U, x^1, \cdots, x^n\}$ be a local chart on X, then any 1-form on U can be written as $\xi = \xi^1 dx^1 + \cdots + \xi^n dx^n$. This gives a local chart $\{T^*U, x^1, \cdots, x^n, \xi^1, \cdots, \xi^n\}$ on T^*X. We consider the following 1-form defined on T^*U:

$$\alpha = \sum_{i=1}^n \xi^i dx^i \in \Lambda^1 T^*(T^*U).$$

Fact: α is independent of the choice of local coordinates.

To see this, we let $y^1, \cdots, y^n$ be another chart on U, then any 1-form can be written as $\sum \eta^i dy^i$. So $\{y^1, \cdots, y^n, \eta^1, \cdots, \eta^n\}$ is another chart on T^*U, and the two charts are related by

$$\eta^i = \sum \xi^j \frac{\partial x^j}{\partial y^i} \quad \text{and} \quad dy^i = \sum \frac{\partial y^i}{\partial x^j} dx^j.$$

It follows $\sum \eta^i dy^i = \sum \xi^i dx^i$.

So we get a global 1-form $\alpha \in \Lambda^1 T^*(T^*X)$. We let

$$\omega = -d\alpha.$$

Then $\omega \in \Lambda^2 T^*(T^*X)$ is a closed 2-form. It is non-degenerate because locally we have

$$\omega = \sum_{i=1}^{n} dx^i \wedge d\xi^i \in \Lambda^2 T^*(T^*U).$$

We will call α the *tautological 1-form* on T^*X, and ω the *canonical symplectic form* on T^*X.