LECTURE 24: DARBOUX THEOREM

1. Moser’s trick

Recall that (PSet 6) an isotopy is a smooth map $\rho : M \times I \to M$ so that the family of maps $\rho_t = \rho(\cdot, t) : M \to M$ is a family of diffeomorphisms with $\rho_0 = \text{Id}$. For each isotopy $\rho_t$ one can construct a time-dependent vector field $X_t$

$$X_t(p) = \left. \frac{d\rho_s}{ds} \right|_{s=t} (\rho_{-t}(p)).$$

The isotopy $\rho_t$ is no longer a flow since in general we no longer have $\rho_t \circ \rho_s = \rho_{t+s}$. However, one can still regard $\rho_t$ as the “flow” generated by the time-dependent vector field $X_t$, since if we let $\gamma(t) = \rho_t(p)$, then we still have

$$\dot{\gamma}(t) = X_t(\gamma(t)).$$

Most results on vector fields and flows (except for the group law alluded above) can be generalized to this “time-dependent” setting. For example, given any compactly-supported time dependent vector field $X_t$, one can construct an isotopy $\rho_t$ so that the previous relation holds. When $X_t$ is not compactly-supported, such a “flow” still exists locally near each point.

In lecture 19 we have seen that for the flow $\rho_t$ generated by a vector field $X$, and for any $k$-form $\alpha$, one has $\frac{d}{dt} \rho^*_t \alpha = \rho^*_t \mathcal{L}_X \alpha$. Using that we proved the homotopy formula

$$(1) \quad \rho^*_t \alpha - \alpha = dQ(\alpha) + Q(d\alpha),$$

where $Q$ is a map $Q : \Omega^k(M) \to \Omega^{k-1}(M)$. In PSet 6 we have seen that in this time-dependent setting, i.e. for isotopy $\rho_t$ we still have

$$(2) \quad \frac{d}{dt} \rho^*_t \alpha = \rho^*_t \mathcal{L}_X \alpha.$$

By repeating the proof in lecture 19 we see that, as long as the isotopy $\rho_t$ exists for $0 \leq t \leq 1$, one still have the homotopy formula (1).

Another consequence of (2) is

Proposition 1.1. Let $\rho_t$ be an isotopy, and $\alpha_t$ a smooth family of $k$-forms. Then

$$(3) \quad \frac{d}{dt} \rho^*_t \alpha_t = \rho^*_t \left( \mathcal{L}_X \alpha_t + \frac{d\alpha_t}{dt} \right).$$

Proof. According to chain rule, for any smooth function $f(x, y)$ of two variables,

$$\frac{d}{dt} f(t, t) = \left. \frac{df}{dx} \right|_{x=t} f(x, t) + \left. \frac{df}{dy} \right|_{y=t} f(t, y).$$
Applying this to our case, we get
\[
\frac{d}{dt}\rho^*_t \alpha_t = \frac{d}{dx} \bigg|_{x=t} \rho^*_t \alpha_t + \frac{d}{dy} \bigg|_{y=t} \rho^*_t \alpha_t = \rho^*_t \left( \mathcal{L}_{X_t} \alpha_t + \frac{d\alpha_t}{dt} \right).
\]

\[\square\]

**Remark.** In what follows we will denote \( \frac{d\alpha_t}{dt} \) by \( \dot{\alpha}(t) \).

Now we are ready to introduce Moser’s trick. The trick was first used by J. Moser in proving theorem 1.2 below in 1965. It turns out that the method works in many situations.

Suppose we have two \( k \)-forms \( \alpha_0 \) and \( \alpha_1 \) on a smooth manifold \( M \) and we are trying to find a diffeomorphism \( \phi : M \to M \) such that \( \phi^* \alpha_1 = \alpha_0 \). Moser’s trick is to construct \( \phi \) as the time-1 flow map of a time-dependent vector field \( X_t \) on \( M \). In fact, Moser’s trick does much more: for a carefully chosen smooth family of \( k \)-forms, \( \alpha_t \), connecting \( \alpha_0 \) and \( \alpha_1 \), one try to find a time-dependent vector field \( X_t \) on \( M \) so that its flow \( \phi_t : M \to M \) satisfies, for all \( 0 \leq t \leq 1 \),

\[\phi^*_t \alpha_t = \alpha_0.\]

(4)

To solve the equation (4), by proposition 1.1 one only need to solve

\[0 = \frac{d}{dt} \phi^*_t \alpha_t = \phi^*_t \left( \dot{\alpha}_t + \mathcal{L}_{X_t} \alpha_t \right).\]

Inserting the Cartan’s magic formula, the equation to be solved becomes

\[\dot{\alpha}_t + \mathcal{L}_{X_t} \alpha_t + \iota_{X_t} d\alpha_t = 0.\]

(5)

The last equation is much easier to solve in many cases.

As an illustration of this method, we prove

**Theorem 1.2** (Moser). Let \( M \) be a compact oriented smooth manifold, and \( \alpha_0, \alpha_1 \) two volume forms on \( M \). Then there exists a diffeomorphism \( \phi : M \to M \) such that \( \phi^* \alpha_1 = \alpha_0 \) if and only if \( \int_M \alpha_0 = \int_M \alpha_1 \).

**Proof.** If such a diffeomorphism exists, then obviously

\[\int_M \alpha_0 = \int_M \phi^* \alpha_1 = \int_{\phi(M)} \alpha_1 = \int_M \alpha_1.\]

Conversely suppose \( \int_M \alpha_0 = \int_M \alpha_1 \), i.e. \( \int_M (\alpha_1 - \alpha_0) = 0 \). Then \( [\alpha_1 - \alpha_0] = 0 \in H^0_{dR}(M) \), i.e. there exists \( \beta \in \Omega^{n-1}(M) \) so that \( \alpha_1 - \alpha_0 = d\beta \).

We take the family of volume forms connecting \( \alpha_0 \) and \( \alpha_1 \) to be

\[\alpha_t = (1 - t)\alpha_0 + t\alpha_1.\]

These \( \alpha_t \)'s are volume forms: Since \( \alpha_0 \) and \( \alpha_1 \) are volume forms, there is a smooth nowhere vanishing function \( f \) on \( M \) so that \( \alpha_1 = f\alpha_0 \). The condition \( \int \alpha_0 = \int \alpha_1 \) implies that \( f > 0 \) everywhere. So \( \alpha_t = [(1 - t) + tf]\alpha_0 \) is a nowhere vanishing top form for \( 0 < t < 1 \).
Moreover by definition, $\dot{\alpha}_t = \alpha_1 - \alpha_0 = d\beta$. We want to find an isotopy $\phi_t$ so that $\phi_t^* \alpha_t = \alpha_0$, which implies the theorem. According to Moser’s trick, it is enough to solve the equation (5), which, in our case, becomes

$$0 = \dot{\alpha}_t + d\iota_{X_t}\alpha_t + \iota_{X_t}d\alpha_t = d(\beta + \iota_{X_t}\alpha_t).$$

This is always solvable, because one can always find a vector field $X_t$ solving the equation

$$\beta + \iota_{X_t}\alpha_t = 0,$$

since $\alpha_t$’s are volume forms. □

Back to symplectic geometry. The following definition is natural:

**Definition 1.3.** Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be symplectic manifolds. A smooth map $f : M_1 \to M_2$ is called a symplectomorphism (or a canonical transformation) if it is a diffeomorphism, and

$$f^* \omega_2 = \omega_1.$$

As usual one would regard symplectomorphic symplectic manifolds as the same.

**Remark.** Suppose we have a diffeomorphism $\varphi : X \to Y$. Then one can “lift” $\varphi$ to a smooth map $\Phi : T^*X \to T^*Y$ via

$$\Phi : T^*X \to T^*Y, \quad (x, \xi) \mapsto (\varphi(x), (d\varphi_{\xi}^*)^{-1}(\xi)).$$

One can prove: $\Phi$ is a symplectomorphism (here we equip $T^*X$ and $T^*Y$ with the canonical symplectic forms we introduced last time).

As an application of Moser’s theorem, we have

**Theorem 1.4** (Classification of compact symplectic surfaces). Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be two closed 2-dimensional symplectic manifolds. Then they are symplectomorphic if and only if they have the same genus and the same symplectic area.

**Proof.** This follows from the fact that two smooth compact oriented surfaces are diffeomorphic if and only if they have the same genus together with Moser’s theorem. □

**Remark.** This is no such classification theorem for symplectic manifolds of dimensions $\geq 4$.

As another application of Moser’s trick, one can prove that a deformation in the same de Rham cohomology class will not give us any new symplectic structure:

**Theorem 1.5.** Let $M$ be compact and $\omega_t = \omega_0 + d\beta_t$ a smooth family of symplectic forms on $M$. Then there exists a smooth family of diffeomorphisms $\phi_t : M \to M$ so that $\phi_t^* \omega_t = \omega_0$.

**Proof.** Repeat Moser’s argument as before. Now Moser’s equation (5) becomes

$$d(\dot{\beta}_t + \iota_{X_t}\omega_t) = 0,$$

and thus it is enough to find a vector field $X_t$ solving

$$\dot{\beta}_t + \iota_{X_t}\omega_t = 0,$$

which is always solvable because of the non-degeneracy of the symplectic forms. □
2. DARBOUX THEOREM

Now we are going to apply Moser’s trick to prove

**Theorem 2.1 (Darboux theorem).** Let \((M, \omega)\) be a symplectic manifold of dimension \(2n\). Then for any \(p \in M\), there exists a neighborhood \(U\) of \(p\) in \(M\) and a neighborhood \(U_0\) of 0 in \(\mathbb{R}^{2n}\) so that \((U, \omega)\) is symplectomorphic to \((U_0, \Omega_0)\), where \(\Omega_0\) is the standard (linear) symplectic form on \(\mathbb{R}^{2n}\).

**Remark.**
(1) In the language of local coordinates, this means that near any \(p \in M\), one can find coordinate patch \((U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)\) centered at \(p\) such that on \(U\),

\[
\omega = \sum dx_i \wedge d\xi_i.
\]

**Definition 2.2.** This coordinate patch is called a **Darboux coordinate patch**.

(2) As a consequence we see that for symplectic manifolds there is no **local geometry**: locally all symplectic manifolds of the same dimension look the same. So symplectic geometry is very different from Riemannian geometry where one has many local geometric quantities (like curvature). (However, there are much to say about the global geometry/topology of symplectic manifolds!)

In fact, we will prove a stronger theorem:

**Theorem 2.3 (Weinstein’s Darboux theorem).** Let \(M\) be a smooth manifold and \(i : N \hookrightarrow M\) a compact submanifold. Let \(\omega_0\) and \(\omega_1\) be two symplectic forms on \(M\) such that \(\omega_0|_N = \omega_1|_N\). Then there exist neighborhoods \(U_0\) and \(U_1\) of \(N\) in \(M\) and a smooth map \(\phi : U_0 \to U_1\) such that \(\phi|_N = \text{Id}\) and \(\phi^* \omega_1 = \omega_0\).

In the proof we need the following variation of Poincare’s lemma whose proof is lefted as an exercise:

**Lemma 2.4.** Let \(i : N \hookrightarrow M\) be a submanifold, \(\alpha \in \Omega^k(M)\) a closed \(k\)-form on \(M\) such that \(i^* \alpha = 0\). Then one can find and a neighborhood \(U\) of \(N\) in \(M\) and a \((k-1)\)-form \(\beta \in \Omega^{k-1}(U)\) with \(\beta = 0\) on \(N\) such that \(\alpha = d\beta\) on \(U\).

**Proof of Weinstein’s Darboux theorem.** Let \(\omega_t = (1 - t)\omega_0 + t\omega_1\). Since \(\omega_t = \omega_0\) on \(N\) and \(N\) is compact, one can find a tubular neighborhood \(U\) of \(N\) so that \(\omega_t\) is symplectic on \(U\) for all \(0 \leq t \leq 1\). According to the lemma above, there exists a 1-form \(\alpha\) on \(U\) with \(\alpha|_N = 0\) such that \(\dot{\omega}_t = \omega_1 - \omega_0 = d\alpha\) on \(U\). Again we solve the equation

\[
\iota_{X_t}\omega_t + \alpha = 0
\]

to get a vector field \(X_t\) on \(U\). Since \(\alpha = 0\) on \(N\), we see \(X_t = 0\) on \(N\). So we may shrink \(U\) to a neighborhood \(U_0\) of \(N\) so that the flow of \(X_t\) is defined for \(0 \leq t \leq 1\) on \(U_0\). Now set \(\phi\) to be the time-1 map of the flow of \(X_t\) on \(U_0\) and set \(U_1 = \phi(U_0)\).

Finally we can prove Darboux theorem:
Proof of Darboux Theorem. We need to show the existence of a Darboux coordinate system near each point \( p \in M \). Pick any symplectic basis \( \{ x'_1, \ldots, x'_n, \xi'_1, \ldots, \xi'_n \} \) for the symplectic vector space \((T_p M, \omega_p)\), and extend it to a coordinate system in a neighborhood \( U' \) of \( p \). On \( U' \) one has two symplectic forms: the given one \( \omega_0 = \omega \), and a new one \( \omega_1 = \sum dx'_i \wedge d\xi'_i \). Now apply the previous theorem with \( X = \{ p \} \) and \( M = U' \), we can find neighborhood \( U_0 \) and \( U_1 \) of \( p \) in \( U' \) and a diffeomorphism \( \varphi : U_0 \to U_1 \) so that \( \varphi(p) = p \) and \( \varphi^*(\sum dx'_i \wedge d\xi'_i) = \omega \).

To complete the proof we only need to set \( x_i = \varphi^*(x'_i) \) and \( \xi_i = \varphi^*(\xi'_i) \). \( \square \)

A natural question is: What can we say about global picture of symplectic geometry? Of course any symplectomorphic is an area-preserving diffeomorphism. Is it true that globally a symplectomorphism is as soft as area-preserving diffeomorphisms? The answer is no. For example, we have the following non-squeezing theorem due to Gromov:

**Theorem 2.5 (Gromov).** If there is a symplectomorphism that maps the unit ball \( B^{2n}(1) \) in \( \mathbb{R}^{2n} \) into the cylinder

\[
Z(r) = B^2(r) \times \mathbb{R}^{2n-2} = \{(x, y) \in \mathbb{R}^{2n} \mid (x^1)^2 + (y^1)^2 \leq 1\},
\]

both equipped with the standard symplectic structure \( \omega_0 \), then \( r \geq 1 \).

So the group of symplectomorphisms, \( \text{Symp}(\mathbb{R}^{2n}, \omega_0) \), is much smaller than \( \text{Diff}_{\text{vol}}(\mathbb{R}^{2n}) \), the group of volume-preserving diffeomorphisms.

**Remark.** However, for any \( \delta > 0 \), the map

\[
(x, y) \mapsto (\delta x, \delta^{-1} y)
\]

is a symplectomorphism on \( \mathbb{R}^{2n} \) which embeds \( B^{2n}(1) \) into \( B^n(r) \times \mathbb{R}^n \).