LECTURE 25: CONNECTIONS AND CURVATURES

1. Connections and curvatures in vector bundles

Recall that a k-form on M is a smooth section of the exterior power bundle $\Lambda^k T^*M$. We can extend the definition to

Definition 1.1. Let E be any vector bundle over M. We call any smooth section of $\Lambda^k T^*M \otimes E$ an *E-valued k-form* on M. The set of all E-valued k-forms is denoted by $\Omega^k(M; E)$.

Of course locally any element $\eta \in \Omega^k(M; E)$ can be written as linear combinations of elements of the form $\omega \otimes s$, where $\omega \in \Omega^k(M)$ and $s \in \Gamma^\infty(E)$. Note that in general one can no longer define the wedge products between two elements in $\Omega^k(M; E)$. However, one can still define a wedge product $\wedge : \Omega^k(M) \times \Omega^l(M; E) \to \Omega^{k+l}(M; E)$ by extending the following rule linearly:

$$\omega_1 \wedge (\omega_2 \otimes s) := (\omega_1 \wedge \omega_2) \otimes s.$$

Similarly one can define $\wedge : \Omega^l(M; E) \otimes \Omega^k(M) \to \Omega^{k+l}(M; E)$. Thus $\Omega^*(M; E)$ is a graded module over the graded algebra $\Omega^*(M)$.

Remark. However, if the fiber E_p's are not just vector spaces, but in fact algebras (so that one can "multiply" vectors in each E_p), then one can define the wedge products between elements in $\Omega^k(M; E)$ and $\Omega^l(M; E)$. This is the case, for example, for $\Omega^k(M; \mathrm{End}(E))$:

$$\wedge : \Omega^k(M; \mathrm{End}(E)) \times \Omega^l(M; \mathrm{End}(E)) \to \Omega^{k+l}(M; \mathrm{End}(E)),$$
$$(\omega_1 \otimes s_1) \wedge (\omega_2 \otimes s_2) := (\omega_1 \wedge \omega_2) \otimes (s_1 \circ s_2).$$

Definition 1.2. Let E be a vector bundle over M. A *linear connection* on E is a linear map

$$\nabla : \Omega^0(M; E) = \Gamma^\infty(E) \to \Gamma^\infty(T^*M \otimes E) = \Omega^1(M; E)$$

such that for any $f \in C^\infty(M)$ and any $s \in \Gamma^\infty(E)$, we have

$$\nabla(fs) = df \otimes s + f\nabla s.$$

Example. If $E = M \times \mathbb{R}^r$ is a trivial vector bundle, then any section in $\Gamma^\infty(E)$ is of the form $s = (f_1, \cdots, f_r)$, where each f_i is a smooth function on M. In this case one can define a trivial connection ∇^0 by $\nabla^0(f_1, \cdots, f_r) := (df_1, \cdots, df_r) \in \Gamma^\infty(T^*M \otimes E)$.

Remark. If ∇^0 and ∇^1 are linear connections on E, then for any $\rho \in C^\infty(M)$,

$$\nabla := \rho\nabla^0 + (1-\rho)\nabla^1$$

is again a linear connection. As a consequence, one can easily prove the existence of linear connections by using "trivial connections on trivialization neighborhoods" and partition of unity.

Remark. A connection is not a tensor since it is not $C^\infty(M)$-linear. However, if ∇^0 and ∇^1 are linear connections on E, then one can check that

$$A := \nabla^1 - \nabla^0 : \Gamma^\infty(E) \to \Gamma^\infty(T^*M \otimes E)$$

satisfies $A(fs) = fA(s)$. In other words, A is a tensor:

$$A \in \Gamma^\infty(T^*M \otimes E \otimes E^*) \simeq \Omega^1(M; E \otimes E^*) \simeq \Omega^1(M; \mathrm{End}(E)).$$

Conversely, for any connection ∇^0 and any $A \in \Omega^1(M, \mathrm{End}(E))$ (viewed as a map from $\Gamma^\infty(E)$ to $\Gamma^\infty(T^*M \otimes E)$ as explained above), one can check that $\nabla^0 + A$ is a connection. So

$$\text{The set of linear connections on } E = \nabla^0 + \Omega^1(M; \mathrm{End}(E)).$$

Now let's describe linear connection ∇ on E locally. Let $\{e_1, \cdots, e_r\}$ be a local frame of E near $x \in M$, i.e. for each y in a neighborhood U of x, $\{e_1(y), \cdots, e_r(y)\}$ form a basis of E_y. Then any section of $E|_U$ can be written as [**In what follows we will apply Einstein's summation convention: automatically sum over repeated upper and lower subscripts**]

$$u = u^j e_j.$$

By definition, one has

$$\nabla u = du^j \otimes e_j + u^j \nabla e_j.$$

So ∇ is completely determined by ∇e_j for a local frame $\{e_1, \cdots, e_r\}$.

Next let's assume that U is a local coordinate patch and the corresponding coordinates near x are given by $\{x^1, \cdots, x^n\}$. Then we get a local frame

$$dx^i \otimes e_j, \quad 1 \le i \le n, 1 \le j \le r$$

of $T^*M \otimes E$. As a consequence, there exist functions Γ^j_{il} on U so that

$$\nabla e_l = \Gamma^j_{il} dx^i \otimes e_j.$$

This implies that for any $u = u^j e_j$,

$$\nabla u = du^j \otimes e_j + \Gamma^j_{il} u^l dx^i \otimes e_j.$$

We let Γ be the following $r \times r$ matrix-valued 1-form (i.e. when paired with a vector, you will get a $r \times r$ matrix)

$$\Gamma = \left(\Gamma^j_{il} dx^i\right)_{1 \le j,l \le r} \in \Omega^1(U) \otimes \mathrm{M}(r, \mathbb{R})$$

Then the previous equation can be abbreviated as

$$\nabla u = du + \Gamma u.$$

We will call Γ the *connection 1-form* associated to the given local frame $\{e_1, \cdots, e_r\}$.

Note that the connection 1-form depends on the choice of local frame. Let $\{\tilde e_1, \cdots, \tilde e_r\}$ be another local frame defined on a coordinate patch U near x. Then we can write u in two ways

$$u^j e_j = u = \tilde u^j \tilde e_j$$

Let g be the invertible $r \times r$ matrix so that

$$(\tilde e_1, \cdots, \tilde e_r) = (e_1, \cdots, e_r) g.$$

Then we get

$$\nabla \tilde{e}_l = \widetilde{\Gamma}_{il}^j dx^i \otimes \tilde{e}_j = \widetilde{\Gamma}_{il}^j dx^i \otimes e_s g_j^s = (g_j^s \widetilde{\Gamma}_{il}^j) dx^i \otimes e_s.$$

and

$$\nabla \tilde{e}_l = \nabla(e_j g_l^j) = dg_l^j \otimes e_j + g_l^j \nabla e_l = dg_l^s \otimes e_s + g_l^j \Gamma_{ij}^s dx^i \otimes e_s.$$

Compare the above two formulae we get $g\widetilde{\Gamma} = dg + \Gamma g$, i.e.

$$\widetilde{\Gamma} = g^{-1} dg + g^{-1} \Gamma g.$$

This is the transition rule relating the connection 1-forms in different local frames.

Conversely, one can prove

Proposition 1.3. *Let E be a rank r vector bundle over M and (U_α) an open cover of M consisting of local trivialization charts for E. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ be transition maps of E. Then any collection of matrix-valued 1-forms $\Gamma_\alpha \in \Omega^1(U_\alpha) \otimes \mathrm{M}(r, \mathbb{R})$ satisfying*

$$\Gamma_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} \Gamma_\alpha g_{\alpha\beta}$$

uniquely defines a linear connection on E.

Proof. Exercise. □

Now let E be a vector bundle over M and $\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$ a connection on E. One can extend ∇ to a family of operators

$$\nabla : \Omega^k(M; E) \to \Omega^{k+1}(M; E)$$

by requiring

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s, \qquad \forall \omega \in \Omega^k(U), s \in \Gamma^\infty(E).$$

One can check that these ∇'s are well-defined and satisfy

$$\nabla(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge \nabla \eta, \qquad \forall \omega \in \Omega^k(M), \eta \in \Omega^l(M; E).$$

Example. Let $E = M \times \mathbb{R}^r$ be the trivial bundle and $\nabla = \nabla^0 : \Omega^0(M; E) \to \Omega^1(M; E)$ be the trivial connection as stated above. Then any element of $\Omega^k(M; E)$ is of the form $\eta = (\eta_1, \cdots, \eta_r)$ with $\eta_i \in \Omega^k(M)$, and the map $\nabla : \Omega^k(M; E) \to \Omega^{k+1}(M; E)$ is merely given by

$$\nabla(\eta_1, \cdots, \eta_r) = (d\eta_1, \cdots, d\eta_r) \in \Omega^{k+1}(M; E).$$

More generally, if $\nabla = d + A$ for some $r \times r$-matrix valued 1-form A, then

$$\nabla(\eta_1, \cdots, \eta_r) = (d\eta_1, \cdots, d\eta_r) + A \wedge (\eta_1, \cdots, \eta_r) \in \Omega^{k+1}(M; E).$$

Lemma 1.4. *For any $f \in C^\infty(M)$ and any $\omega \in \Omega^k(M; E)$ one has*

$$\nabla^2(f\omega) = f(\nabla^2 \omega).$$

Proof. We have

$$\nabla^2(f\omega) = \nabla(df \wedge \omega + f\nabla\omega) = -df \wedge \nabla\omega + df \wedge \nabla\omega + f\nabla^2\omega = f\nabla^2\omega.$$

□

In particular, we see that $\nabla^2 : \Omega^0(M;E) = \Gamma^\infty(E) \to \Omega^2(M;E) = \Gamma^\infty(\wedge^2 T^*M \otimes E)$ is a tensor, and in fact is an $r \times r$ matrix valued 2-form:

$$\nabla^2 \in \Gamma^\infty(\wedge^2 T^*M \otimes E \otimes E^*) \simeq \Omega^2(M; E \otimes E^*) \simeq \Omega^2(M; \mathrm{End}(E)).$$

Note that although the matrix-valued 1-form Γ is only locally defined, the matrix-valued 2-form ∇^2 is globally defined.

Definition 1.5. Given any connection ∇ on E, we will call

$$F(\nabla) := \nabla^2 \in \Omega^2(M; \mathrm{End}(E))$$

the *curvature* of ∇.

Example. Again consider the trivial bundle $E = M \times \mathbb{R}^r$. Let $\nabla = d + A$ be any linear connection on E, where A is any $r \times r$-valued 1-form on M. Then the curvature of ∇ is the two form such that for any $u = (f_1, \cdots, f_r)$,

$$F(\nabla)u = \nabla(du + Au) = A \wedge du + d(Au) + A \wedge Au = (dA + A \wedge A)u.$$

In other words,

$$F(\nabla) = dA + A \wedge A.$$

Note that in the above example, we have $F(\nabla^0) = 0$.

Definition 1.6. A connection ∇ on E is called *flat* if $F(\nabla) = 0$.

2. Connections and curvatures in principal bundles

Recall that a principal G-bundle P over M is a fiber bundle whose fiber and structure group are both G. A principle G-bundle can be described by using the following data:

- An open covering $\{U_\alpha\}$ of M
- A collection of transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ satisfying the cocycle condition

$$g_{\gamma\alpha}(m) = g_{\gamma\beta}(m) g_{\beta\alpha}(m), \quad \forall m \in U_\alpha \cap U_\beta \cap U_\gamma.$$

For simplicity we will assume G is a linear group below, and as usual we denote its Lie algebra by $\mathfrak{g}$. (The definitions/results below hold for more general Lie groups with slight modifications.) Given any principal G-bundle P, one can construct a vector bundle $\mathrm{Ad}(P)$ to be the vector bundle with fiber $\mathfrak{g}$, and, with transition maps (with respect to the open covering $\{U_\alpha\}$)

$$\mathrm{Ad}(g_{\beta\alpha}) : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathfrak{g}),$$

where recall that Ad is the adjoint action of G on $\mathfrak{g}$ given by

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), \quad g \mapsto (A \mapsto gAg^{-1}).$$

Since any fiber of $\mathrm{Ad}(P)$ is a Lie algebra $\mathfrak{g}$, there is a natural Lie bracket operator between vectors in $\mathfrak{g}$. This induces a bracket operation (or wedge product operation if you prefer) for bundle valued differential forms $[\cdot,\cdot] : \Omega^k(M; \mathrm{Ad}(P)) \times \Omega^l(M; \mathrm{Ad}(P)) \to \Omega^{k+l}(M; \mathrm{Ad}(P))$ such that

$$[\omega \otimes X, \eta \otimes Y] = (\omega \wedge \eta) \otimes [X, Y], \quad \forall \omega \in \Omega^k(M), \eta \in \Omega^l(M) \text{ and } X, Y \in \Gamma^\infty(\mathrm{Ad}(P)).$$

It is easy to see that for any $\omega \in \Omega^k(M; \mathrm{Ad}(P))$, $\eta \in \Omega^l(M; \mathrm{Ad}(P))$ and $\zeta \in \Omega^r(M; \mathrm{Ad}(P))$,

(1) $[\omega,\eta]=-(-1)^{kl}[\eta,\omega]$.
(2) $d[\omega,\eta]=[d\omega,\eta]+(-1)^k[\omega,d\eta]$.
(3) $[[\omega,\eta],\zeta]+(-1)^{k(l+r)}[[\eta,\zeta],\omega]+(-1)^{(k+l)r}[[\zeta,\omega],\eta]=0$

Remark. Let $G=\mathrm{GL}(r,\mathbb{R})$ and thus $\mathfrak{g}=\mathfrak{gl}(r,\mathbb{R})=\mathrm{End}(\mathbb{R}^r)$. Let A be any $r\times r$-matrix valued 1-form on M. Then

$$A\wedge A=\frac{1}{2}[A,A].$$

To see this, one just write $A=\omega_i^j\otimes E_j^i$, where E_j^i is the matrix with value 1 at the (i,j)-entry and 0 at other entries, and $\omega_i^j\in\Omega^1(M)$. Then

$$A\wedge A=(\omega_i^j\otimes E_j^i)\wedge(\omega_r^s\otimes E_s^r)=(\omega_i^j\wedge\omega_j^s)\otimes E_s^i,$$

while (use $[E_j^i,E_s^r]=E_j^iE_s^r-E_s^rE_j^i=E_s^i\delta_j^r-E_j^r\delta_s^i$)

$$[A,A]=[\omega_i^j\otimes E_j^i,\omega_r^s\otimes E_s^r]=(\omega_i^j\wedge\omega_r^s)\otimes(E_s^i\delta_j^r-E_j^r\delta_s^i)=(\omega_i^j\wedge\omega_j^s)\otimes E_s^i+(\omega_r^s\wedge\omega_s^j)\otimes E_j^r=2A\wedge A.$$

(More generally one can prove $A\wedge B=\frac{1}{2}[A,B]$ for any $r\times r$-matrix valued 1-forms A and B.)

Definition 2.1. Let P be a principal G-bundle over M defined by an open covering $\{U_\alpha\}$ and transition maps $g_{\beta\alpha}:U_\alpha\cap U_\beta\to G$.

(1) A *connection* A on P is a collection $A_\alpha\in\Omega^1(U_\alpha)\otimes\mathfrak{g}=\Omega^1(U_\alpha;\mathrm{Ad}(P)|_{U_\alpha})$ such that
$$A_\beta(m)=g_{\alpha\beta}^{-1}(m)dg_{\alpha\beta}(m)+g_{\alpha\beta}^{-1}(m)A_\alpha(m)g_{\alpha\beta}(m),\qquad\forall m\in M.$$
(2) The *curvature* of a connection A on P is the collection $F_\alpha\in\Omega^2(U_\alpha)\otimes\mathfrak{g}$, where
$$F_\alpha=dA_\alpha+\frac{1}{2}[A_\alpha,A_\alpha].$$

By definition the collection $\{A_\alpha\}$ is not a global object defined on M. But we have

Proposition 2.2. *Let $\{A_\alpha\}$ be a connection on P. Then $\{B_\alpha\}$ is a connection on P if and only if the collection $\{C_\alpha=A_\alpha-B_\alpha\}$ defines an element in $\Omega^1(M;\mathrm{Ad}(P))$.*

Proof. If both $\{A_\alpha\}$ and $\{B_\alpha\}$ are connections on P, then by definition

$$C_\beta=g_{\beta\alpha}C_\alpha g_{\beta\alpha}^{-1}=\mathrm{Ad}_{g_{\beta\alpha}}C_\alpha.$$

Since $\mathrm{Ad}_{g_{\beta\alpha}}$'s are the transition maps for $\mathrm{Ad}(P)$, the collection $\{C_\alpha\}$ define an element in $\Omega^1(M;\mathrm{Ad}(P))$.

Conversely if $\{A_\alpha\}$ is a connection and $C\in\Omega^1(M;\mathrm{Ad}(P))$, then it is easy to check that the collection $\{B_\alpha=A_\alpha+C|_{U_\alpha}\}$ define a connection on P. □

Remark. By using partition of unity one can prove the existence of a connection on any principal G-bundle. As a consequence, the set of all connections on P is an affine space which is a "translation copy" of $\Omega^1(M;\mathrm{Ad}(P))$.

Again we have

Proposition 2.3. *The collection $\{F_\alpha\}$ defines an element $F(A)$ in $\Omega^2(M;\mathrm{Ad}(P))$.*

Proof. We need to show that the collection $\{F_\alpha\}$ satisfies

$$F_\beta = g_{\beta\alpha} F_\alpha g_{\beta\alpha}^{-1}.$$

First notice that for any matrix-valued function g, we have the so-called *Maurer-Cartan equations*

$$d(g^{-1}dg) = dg^{-1} \wedge dg = -g^{-1}(dg)g^{-1} \wedge dg = -(g^{-1}dg) \wedge (g^{-1}dg) = -\frac{1}{2}[g^{-1}dg, g^{-1}dg].$$

Also for any matrix-valued 1-form A, one has

$$\begin{aligned} d(g^{-1}Ag) &= -g^{-1}(dg)g^{-1} \wedge Ag + g^{-1}(dA)g - g^{-1}A \wedge dg \\ &= -g^{-1}dg \wedge g^{-1}Ag + g^{-1}(dA)g - g^{-1}Ag \wedge g^{-1}dg \\ &= -\frac{1}{2}[g^{-1}dg, g^{-1}Ag] + g^{-1}(dA)g - \frac{1}{2}[g^{-1}Ag, g^{-1}dg] \\ &= -[g^{-1}dg, g^{-1}Ag] + g^{-1}(dA)g. \end{aligned}$$

As a consequence, we see (with $g = g_{\alpha\beta}$ for abbreviation)

$$\begin{aligned} F_\beta &= dA_\beta + \frac{1}{2}[A_\beta, A_\beta] \\ &= d(g^{-1}dg + g^{-1}A_\alpha g) + \frac{1}{2}[g^{-1}dg + g^{-1}A_\alpha g, g^{-1}dg + g^{-1}A_\alpha g] \\ &= g^{-1}(dA_\alpha)g + \frac{1}{2}[g^{-1}A_\alpha g, g^{-1}A_\alpha g] \\ &= g^{-1}F_\alpha g. \end{aligned}$$

□

Next we will prove the Bianchi identity:

Proposition 2.4 (The Bianchi identity)**.** *Notations as above. One has*

$$dF_\alpha + [A_\alpha, F_\alpha] = 0.$$

Proof. This follows from a direct computation:

$$dF_\alpha = \frac{1}{2}[dA_\alpha, A_\alpha] - \frac{1}{2}[A_\alpha, dA_\alpha] = [dA_\alpha, A_\alpha] = [F_\alpha - \frac{1}{2}[A_\alpha, A_\alpha], A_\alpha] = [F_\alpha, A_\alpha],$$

where in the last step we used the fact that for any matrix-valued 1-form A,

$$[[A, A], A] + [[A, A], A] - [[A, A], A] = 0 \Longrightarrow [[A, A], A] = 0.$$

□