LECTURE 26: THE CHERN-WEIL THEORY

1. Invariant Polynomials

We start with some necessary backgrounds on invariant polynomials. Let V be a vector space. Recall that a k-tensor $T \in \otimes^k V^*$ is called *symmetric* if

$$T(v_{\sigma_1}, \cdots, v_{\sigma_k}) = T(v_1, \cdots, v_k), \qquad \forall \sigma \in S_k.$$

We will denote the space of all symmetric k-tensors on V by $S^k V^*$.

Let $T \in S^k V^*$ be any symmetric k-tensor. Then T induces a "degree k homogeneous polynomial on V",

$$P_T(v) := T(v, \cdots, v).$$

Conversely, it is not hard to see that T is completely determined by P_T, since we have the following *polarization formula*

$$T(v_1, \cdots, v_k) := \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k} P_T(t_1 v_1 + \cdots + t_k v_k).$$

Like wedge product, we can define a *symmetric product* $\circ : S^k V^* \times S^l V^* \to S^{k+l} V^*$ via

$$T_1 \circ T_2(v_1, \cdots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} T_1(v_1, \cdots, v_k) T_2(v_{k+1}, \cdots, v_{k+l}).$$

As usual we put $S^0 V^* = \mathbb{R}$.

Of course P_T defined as above is not really a polynomial. (At least one should need a conception of "products of vectors" on V before we can talk about polynomials on V.) However, we can associate to each $T \in S^k V^*$ a true polynomial as follows: We fix a basis $\{e_1, \cdots, e_n\}$ of V. Let $\mathbb{R}[x_1, \cdots, x_n]$ be the ring of polynomials in the variable $x_1, \cdots, x_n$, and let $\mathbb{R}[x_1, \cdots, x_n]^k$ be the subset of homogeneous polynomials of degree k. We define a map $\mathcal{P} : S^k V^* \to \mathbb{R}[x_1, \cdots, x_n]^k$ by

$$T \mapsto p_T(x_1, \cdots, x_n) := P_T\left(\sum x_i e_i\right) = T\left(\sum x_i e_i, \cdots, \sum x_i e_i\right)$$

Obviously $\mathcal{P}$ is a linear map and satisfies $\mathcal{P}(T_1 \circ T_2) = \mathcal{P}(T_1)\mathcal{P}(T_2)$.

Lemma 1.1. *The map $\mathcal{P}$ is a linear isomorphism. (In particular,* $\dim S^k V^* = \binom{n+k-1}{n-1}$*.)*

Proof. By the polarization formula above we see $\mathcal{P}$ is injective. To show that $\mathcal{P}$ is surjective, we just notice that for $k = 1$, $\mathcal{P}$ maps $T_i \in S^1 V^* = V^*$ to the polynomial $p_{T_i}(x_1, \cdots, x_n) = x_i$, where T_i is the map $T_i(\sum x_i e_i) := x_i$. For general k, by repeatedly using the rule $\mathcal{P}(T_1 \circ T_2) = \mathcal{P}(T_1)\mathcal{P}(T_2)$ one immediately see that $\mathcal{P}$ is surjective. $\square$

Remark. The fact $\mathcal{P}(T_1 \circ T_2) = \mathcal{P}(T_1)\mathcal{P}(T_2)$ implies that $\mathcal{P} : S^* V^* \to \mathbb{R}[x_1, \cdots, x_n]$ is in fact a ring isomorphism.

Now let $V=\mathfrak{g}$ be the Lie algebra of a Lie group G. Then the adjoint action of G on $\mathfrak{g}$ induces a G-action on $S^k(\mathfrak{g}^*)$ by

$$(g\cdot T)(X_1,\cdots,X_k):=T(\mathrm{Ad}_{g^{-1}}X_1,\cdots,\mathrm{Ad}_{g^{-1}}X_k),\qquad \forall X_i\in\mathfrak{g},g\in G.$$

Note that for the case that G is a linear Lie group, which we will always assume below, one has

$$(g\cdot T)(X_1,\cdots,X_k)=T(gX_1g^{-1},\cdots,gX_kg^{-1}),$$

where g is an invertible $r\times r$ matrix while X_i's are arbitrary $r\times r$ matrices.

Definition 1.2. $T\in S^k(\mathfrak{g}^*)$ is called *invariant* if

$$g\cdot T=T,\qquad \forall g\in G.$$

The set of all G-invariant elements in $S^k(\mathfrak{g}^*)$ is denoted by $I^k(G)$.

So $I^*(G)=\oplus I^k(G)$ is a subring of $S^*(\mathfrak{g}^*)$. Note that by definition and the polarization formula, T is invariant if and only if P_T is invariant.

Example. Consider $G=\mathrm{GL}(r,\mathbb{R})$. For any positive integer k, we let $p_{k/2}$ denote the degree k homogeneous polynomial which is the coefficient of λ^{r-k} for the following polynomial in λ

$$\det\left(\lambda I-\frac{1}{2\pi}A\right)=\sum_{k=0}^{r}p_{k/2}(A,\cdots,A)\lambda^{r-k},\qquad \forall A\in\mathfrak{gl}(n,\mathbb{R}).$$

Obviously $p_{k/2}$ is invariant. So $p_{k/2}\in I^k(\mathrm{GL}(r,\mathbb{R}))$. They are called the $k/2$-th *Pontrjagin polynomials.*

Note that for $G=O(r)\subset \mathrm{GL}(r,\mathbb{R})$, the Lie algebra $\mathfrak{o}(r)$ consists of skew-symmetric matrices, and hence

$$\det\left(\lambda I-\frac{1}{2\pi}A\right)=\det\left(\lambda I+\frac{1}{2\pi}A\right),\qquad \forall A\in\mathfrak{o}(r).$$

This implies that $p_{k/2}=0$ for k odd. So for $O(n)$ one only need to study $p_k\in I^{2k}(\mathrm{O}(n))$.

Proposition 1.3. *For any $T\in I^k(G)$ and any $X,X_1,\cdots,X_k\in\mathfrak{g}$, we have*

$$T([X,X_1],X_2,\cdots,X_k)+\cdots+T(X_1,\cdots,X_{k-1},[X,X_k])=0.$$

Proof. This follows from

$$\left.\frac{d}{dt}\right|_{t=0}T(e^{tX}X_1e^{-tX},\cdots,e^{tX}X_ke^{-tX})=0.$$

$\square$

For $T\in I^k(G)$ one can also define $T(F_1,\cdots,F_k)$ for $F_i\in\Omega^*(U)\otimes\mathfrak{g}$, by extending linearly the relation

$$T(\omega_1\otimes X_1,\cdots,\omega_k\otimes X_k):=(\omega_1\wedge\cdots\wedge\omega_k)T(X_1,\cdots,X_k).$$

Note that if ω is a 2-forms, or more generally is any even-form, then for any η one has $\omega\wedge\eta=\eta\wedge\omega$. As a consequence we see

Corollary 1.4. *If $F_1, \cdots, F_k \in \Omega^{\text{even}}(U) \otimes \mathfrak{g}$, then for any $T \in I^k(G)$ and $A \in \Omega^*(U) \otimes \mathfrak{g}$ one has*
$$T([A, F_1], F_2, \cdots, F_k) + \cdots + T(F_1, \cdots, F_{k-1}, [A, F_k]) = 0.$$

2. Chern-Weil Theory

Let G be a Lie group (and for simplicity, assume G is a linear Lie group), with Lie algebra $\mathfrak{g}$. Let P be a principal G-bundle over M, which is defined by an open cover $\{U_\alpha)$ and corresponding transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

Let A be a collection $A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$ which is a connection in P. Recall that the curvature F_A of the connection A is given locally by the collection
$$F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha] \in \Omega^2(U_\alpha) \otimes \mathfrak{g}.$$
For any $T \in I^k(G)$ we can define
$$p_T(F_\alpha) := T(F_\alpha, \cdots, F_\alpha) \in \Omega^{2k}(U_\alpha).$$
Since T is Ad-invariant and since $F_\beta = g_{\beta\alpha} F_\alpha g_{\beta\alpha}^{-1}$, we get
$$p_T(F_\alpha) = p_T(F_\beta) \qquad \text{on} \quad U_\alpha \cap U_\beta.$$
In other words, there is a globally-defined $2k$-form $p_T(F_A) \in \Omega^{2k}(M)$ so that $p_T(F_A) = p_T(F_\alpha)$ on U_α. (Note F_A is not an element in $\Omega^2(M)$: it sits in $\Omega^2(M; \mathrm{Ad}(P))$.)

It turns out that these $p_T(F_A)$'s play a crucial role in modern math and physics.

Theorem 2.1 (Chern-Weil)**.** *Let P be a principal G-bundle over M. Then*

1. *For any $T \in I^k(G)$ and any connection A in P, the form $p_T(F_A)$ is closed.*
2. *The de Rham class $[p_T(F_A)] \in H^{2k}_{dR}(M)$ is independent of the choices of A.*
3. *The Chern-Weil map $\mathcal{CW} : (I^*(G), \circ) \to (H^*_{dR}(M), \wedge)$ that maps T to $[p_T(F_A)]$ is a ring homomorphism.*

Proof. (1) According to the Bianchi identity $dF_\alpha = -[A_\alpha, F_\alpha]$. So by corollary 1.4,
$$dp_T(F_A) = dT(F_\alpha, \cdots, F_\alpha) = T(dF_\alpha, \cdots, F_\alpha) + \cdots + T(F_\alpha, \cdots, dF_\alpha) = 0.$$
This proves $p_T(F_A)$ is closed.

(2) To prove that the de Rham class $[p_T(F_A)]$ is independent of the choices of connection A, we let A_0 and A_1 be two connections defined by the collections A^0_α and A^1_α. By definition it is easy to check that the collection
$$\widetilde{A}_\alpha = (1 - s)A^0_\alpha + sA^1_\alpha \in \Omega^1(U_\alpha \times \mathbb{R}) \otimes \mathfrak{g}$$
defines a connection on a new principal bundle $P \times \mathbb{R}$ over $M \times \mathbb{R}$ (This new principal bundle is defined over open sets $\{U_\alpha \times \mathbb{R}\}$ by using the same set of transition functions $g_{\alpha\beta}$). If we let ι_1 and ι_0 be the same as in the following lemma, then $\iota_0^* \widetilde{A}_\alpha = A^0_\alpha$ and $\iota_1^* \widetilde{A}_\alpha = A^1_\alpha$. As a result, we see
$$\iota_0^* F_{\widetilde{A}} = F_{A^0} \qquad \text{and} \qquad \iota_1^* F_{\widetilde{A}} = F_{A^1}.$$

So according to the next lemma,

$$p_T(F_{A^0}) - p_T(F_{A^1}) = \iota_0^* p_T(F_{\widetilde{A}}) - \iota_1^* p_T(F_{\widetilde{A}}) = dQ(p_T(F_{\widetilde{A}})) + Qd(p_T(F_{\widetilde{A}})).$$

But we just proved that $p_T(F_{\widetilde{A}})$ is closed. So $[p_T(F_{A^0})] = [p_T(F_{A^1})]$.

Lemma 2.2. *Let $\iota_0, \iota_1 : M \to M \times \mathbb{R}$ be the inclusions*

$$\iota_0(x) = (x, 0), \qquad \iota_1(x) = (x, 1).$$

Then there exists a collection of linear operators $Q : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M)$ so that

$$\iota_0^*\omega - \iota_1^*\omega = dQ(\omega) - Qd(\omega), \qquad \forall \omega \in \Omega^k(M \times \mathbb{R}).$$

Proof. One just repeat the first four lines of the proof of theorem 2.6 in lecture 19, to get some linear map $\widetilde{Q} : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M \times \mathbb{R})$ so that $\omega - \phi_1^*\omega = d\widetilde{Q}\omega + \widetilde{Q}d\omega$. It follows that

$$\iota_0^*\omega - \iota_1^*\omega = \iota_0^*\omega - \iota_0^*\phi_1^*\omega = \iota_0^* d\widetilde{Q}\omega + \iota_0^*\widetilde{Q}d\omega = d(\iota_0^*\widetilde{Q})\omega + (\iota_0^*\widetilde{Q})d\omega.$$

So the conclusion holds for $Q = \iota_0^*\widetilde{Q} : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M)$. □

(3) It is straightforward to show that for $T \in S^k(G)$ and $S \in S^l(G)$, $p_{T\circ S}(F_A) = p_T(F_A) \wedge p_S(F_A)$. Details left as an exercise. □

Example. Consider the trivial principal G-bundle $P = M \times G$ over M. In this case one can take $U_\alpha = M$, i.e. the open cover contains only one element. Then there is only one $g_{\alpha\alpha}$, which equals $e \in G$ at each point $x \in M$. For such a covering data one can take A_α to be identically zero. It follows that $F_A = 0$ and thus for any $T \in I^*(G)$,the class $[p_T(F_A)] = 0$.

Definition 2.3. For any $T \in I^*(G)$, the cohomology class $[p_T(F_A)]]$ is called the *characteristic class* for P corresponding to T.

Remark. Obviously the above construction works if we replace the principal bundle P by a vector bundle E. So one can also talk about the charactersitic classes of vector bundle E over M associated with $T \in I^*(G)$, where G is the structural group of E.

Example. Let E be any vector bundle over M. By choosing a Riemannian metric one can always reduce the structural group of E from $\mathrm{GL}(r, \mathbb{R})$ to $\mathrm{O}(r)$. Let p_k be the Pontrjagin polynomial that we alluded above. Then

$$p_k(E) := [p_k(F_A)] \in H^{4k}_{dR}(M)$$

is called the kth *Pontrjagin characteristic class* of E. The Pontrjagin class $p_k(M)$ of M is defined to be the Pontrjagin class of TM. They are important topological invariants to study manifolds.

Example. Suppose $r = 2p$ and consider $G = \mathrm{SO}(r)$. For any $A = (a^i_j) \in \mathfrak{so}(r)$ we let

$$Pf(A) = \frac{1}{(4\pi)^r (r/2)!} \sum_{\sigma \in S_r} (-1)^\sigma a^{\sigma(1)}_{\sigma(2)} a^{\sigma(3)}_{\sigma(4)} \cdots a^{\sigma(r-1)}_{\sigma(r)}.$$

One can check that Pf is an Ad-invariant homogeneous polynomial of degree $r/2$. It is called the *Pfaff polynomial*.

Now suppose E is an oriented vector bundle over M of rank r. Then the structural group of E can be reduced to $\mathrm{SO}(r)$. Thus we get a characteristic class

$$e(E) := [Pf(F_A)] \in H^r_{dR}(M)$$

which is called the *Euler characterstic class* of E. Again the Euler class $e(M)$ of a smooth manifold M is defined to be the Euler class of TM. It generalizes the classical notion of Euler characteristic.

Example. All discusstions above are over $\mathbb{R}$, but they can be generalized to objects over $\mathbb{C}$. For example, one can talk about *complex vector bundle* E over smooth manifold M: They are vector bundles over M whose fibers are $\mathbb{C}^r$ and whose structural group is $\mathrm{GL}(r, \mathbb{C})$. For any $A \in \mathfrak{gl}(r, \mathbb{C})$ (= the set of all $r \times r$ complex matrices), one can define c_k to be such that

$$\det\left(\lambda I_r - \frac{1}{2\pi i}A\right) = \sum c_k(A)\lambda^{n-k}.$$

Again c_k is a homogeneous polynomial of degree k and is Ad-invariant. As a result, for any complex vector bundle E of complex rank r, one gets a complex-valued de Rham cohomology class

$$c_k(E) := c_k(F_A) \in H^{2k}_{dR}(M; \mathbb{C})$$

which is called the kth *Chern class* of E. If one fix an Hermitian metric on E (i.e. fix a Hermitian inner product on each fiber of E which vary smoothly with respect to base points – this is always possible by using partition of unity), then one can reduce the structural group to $U(n)$. But for $A \in \mathfrak{u}(n)$ (= the set of all $r \times r$ complex matrices with $A + \overline{A}^T = 0$), one has

$$\det\left(\lambda I_r - \frac{1}{2\pi i}A\right) = \overline{\det\left(\bar{\lambda} I_r - \frac{1}{2\pi i}A\right)}.$$

It follows that each c_k is a real-valued polynomial and thus each $c_k(E) \in H^{2k}_{dR}(M)$, i.e. $c_k(E)$ is a (real-valued) de Rham cohomology class. It is of fundamental importance in algebraic topology, differential geometry, algebraic geometry and mathematical physics.

—The End—