LECTURE 1: THE RIEMANNIAN METRIC

0. Course Information

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Instructor: Zuoqin Wang
Email: wangzuoq@ustc.edu.cn

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Lecture time/room: Monday 9:45-11:20 AND Saturday 19:30-21:05 @ 5505
Course webpage: http://staff.ustc.edu.cn/~wangzuoq/Courses/16S-RiemGeom/

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Course grades = exercises (40%) + midterm exam(30%) + final essay (30%)

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Reference books (not required. Course notes will be posted online.)

- *Riemannian geometry* by de Carmo
- *Riemannian geometry* by P. Peterson

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Rough plan of this course:

- Basic Riemannian geometry I: metric, connection and curvature
  - Riemannian metric, distance, length, volume
  - Connections, parallel transport, Levi-Civita
  - Sectional, Ricci and scalar curvature
  - Special classes of Riemannian manifolds
- Basic Riemannian Geometry II: geodesics
  - Exponential map, normal coordinates
  - Jacobi field, variational formulae
  - Index form, Morse index theorem
- Topology of Riemannian manifolds
  - Cartan-Hardamard, Bonnet-Myers, Synge
  - Comparison theorems: Rauch, Toponogov, volume etc
  - Critical point theory of distance function
- Analysis on Riemannian manifolds (if time permits)
  - Weitzenbock, towards Hodge theory
  - Laplace operator: eigenvalues, eigenfunctions
Einstein Summation Convention: If in a term the same index appears twice, both as an upper and a lower index, that term is assumed to be summed over all possible values of that index (usually from 1 to the dimension). For example,

\[ a^i b^i := \sum_i a^i b^i, \quad a^{ijkl} b_{il} c^j := \sum_{i,j,l} a^{ijkl} b_{il} c^j. \]

1. The Riemannian metric

Let \( M \) be a smooth manifold of dimension \( m \). Recall that this implies locally near every point \( p \in M \) there is a neighborhood \( U \) of \( p \) which is diffeomorphic to a domain in \( \mathbb{R}^m \). Moreover, if we denote by \( \{ x^1, \cdots, x^m \} \) the coordinate functions on \( U \), then the tangent space \( T_p M \) is spanned by the vectors \( \{ \partial_1, \cdots, \partial_m \} \), and its dual \( T^*_p M \) is spanned by \( \{ dx^1, \cdots, dx^m \} \).

**Definition 1.1.** A Riemannian metric \( g \) on \( M \) is an assignment of an inner product \( g_p(\cdot, \cdot) = \langle \cdot, \cdot \rangle_p \) on \( T_p M \) for each \( p \in M \) that depends smoothly on \( p \).

**Remarks.**
1. “Smooth dependence” \( \iff \) if \( X, Y \) are two smooth vector fields on an open subset \( U \subset M \), then \( f(p) = \langle X_p, Y_p \rangle_p \) is a smooth function on \( U \).
2. \( g \) itself is NOT a metric (aka a distance function) on \( M \). Recall that a distance function on \( M \) is a continuous function \( d : M \times M \to \mathbb{R} \) so that for all \( p, q, r \in M \),
   - \( d(p, q) \geq 0 \) and \( d(p, q) = 0 \) if and only if \( p = q \);
   - \( d(p, q) = d(q, p) \);
   - \( d(p, r) \leq d(p, q) + d(q, r) \).

However, we will see soon that \( g \) induces a natural distance function \( d \) on \( M \).

One can represent the Riemannian metric \( g \) using local coordinates as follows. Let \( \{ U, x^1, \cdots, x^m \} \) be a coordinate patch. We denote

\[ g_{ij}(p) = \langle \partial_i, \partial_j \rangle_p. \]

Then for any smooth vector fields \( X = X^i \partial_i \) and \( Y = Y^j \partial_j \) in \( U \),

\[ \langle X_p, Y_p \rangle_p = X^i(p)Y^j(p)\langle \partial_i, \partial_j \rangle_p = g_{ij}(p)X^i(p)Y^j(p). \]

So locally we can write

\[ g = g_{ij} dx^i \otimes dx^j. \]

It is easy to see that the coefficients \( g_{ij} \) have the following properties:
- For all \( i, j \), \( g_{ij}(p) \) is smooth in \( p \).
- \( g_{ij} = g_{ji} \), so the matrix \((g_{ij}(p))\) is symmetric at any \( p \).
- The matrix \((g_{ij}(p))\) is also positive definite for any \( p \).

This gives another description of a Riemannian metric \( g \):

A Riemannian metric \( g \) is a smooth symmetric \((0, 2)\)-tensor field that is positive definite everywhere.
Note that each matrix \((g_{ij})\) is positive definite. We will denote by \((g^{ij})\) the inverse matrix of \((g_{ij})\), i.e. they satisfy
\[
g_{ij}g^{jk} = \delta^k_i.
\]
Then the matrix \((g^{ij})\) is again positive definite, and we use it to define a dual inner product structure on \(T^*_p M\) for each \(p\). More explicitly, for any 1-forms \(\omega = \omega_i dx^i\) and \(\eta = \eta_j dx^j\) on \(U\),
\[
\langle \omega, \eta \rangle_p := g^{ij}(p)\omega_i(p)\eta_j(p).
\]
Since \(g\) is non-degenerate and bilinear on \(T_p M\), it gives us an isomorphism between \(TM\) and \(T^* M\) via
\[
b : TM \to T^* M, \quad b(X)(Y) := g(X, Y).
\]
(Pronunciation of \(b\): flat)
It is not hard to check that \(b\) is a vector bundle isomorphism. In local coordinates, if we denote \(X = X^i \partial_i\) and take \(Y = \partial_j\) for each \(j\), then
\[
b(X)(\partial_j) = g(X, \partial_j) = g_{ij}X^i,
\]
so we conclude
\[
b(X^i \partial_i) = g_{ij}X^i dx^j.
\]
We will denote the inverse map of \(b\) by
\[
\sharp : T^* M \to TM.
\]
(Pronunciation of \(\sharp\): sharp)
Locally
\[
\sharp(w_i dx^i) = g^{ij}w_i \partial_j.
\]
Note that \(b\) is “lowering the indeces”, i.e. change the coefficient from \(X^i\) to \(g_{ij}X^i\), using \(g_{ij}\), while \(\sharp\) is “raising the indeces” via \(g^{ij}\). We will call \(b\) and \(\sharp\) the musical isomorphisms. [In music, the symbol \(\flat\) means lower in pitch while the symbol \(\sharp\) means higher in pitch.] Note that they are actually defined pointwise, and the dual inner product on \(T^*_p M\) we mentioned above is merely
\[
\langle \omega, \eta \rangle_p := g_p(\sharp^* \omega, \sharp^* \eta).
\]
Remark. More generally, the Riemannian inner product \(g\) induced an natural inner product \(\tilde{g}\) on \((T_p M)^{\otimes k} \otimes (T^*_p M)^{\otimes l}\). Suppose \(e_1, \ldots, e_m\) is an orthonormal basis of \(T_p M\), and \(e^1, \ldots, e^m\) its dual basis. Then the induced inner product is defined so that
\[
\{e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l}\}
\]
form an orthonormal basis.
2. Riemannian manifolds

Let $M$ be a smooth manifold.

**Definition 2.1.** Let $g$ be Riemannian metric on $M$. Then we call the pair $(M, g)$ a **Riemannian manifold**.

[Note: Sometimes we omit $g$ and say $M$ is a Riemannian manifold.]

The simplest example of Riemannian manifold is

**Example.** The standard inner product on $\mathbb{R}^m$ defines a **canonical Riemannian metric** $g_0$ on $\mathbb{R}^m$ via

$$g_0(X,Y) = \sum_i X^i Y^i.$$  

Alternatively, this means the matrix $(g_{ij})$ is the identity matrix:

$$(g_0)_{ij} = \delta_{ij}.$$ 

In the notion of tensors, we can write

$$g_0 = dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m.$$ 

More generally, for any positive definite $m \times m$ matrix $A = (a_{ij})$, the formula

$$g^A_p(X_p, Y_p) := X_p^T A Y_p$$

defines a Riemannian metric on $\mathbb{R}^m$ in which case $g^A_{ij} = a_{ij}$. Equivalently,

$$g^A = \sum_{i,j} a_{ij} dx^i \otimes dx^j.$$ 

There are many ways to construct new Riemannian manifolds from old, for example,

1. Let $(M, g_M)$ and $(N, g_N)$ be two Riemannian manifolds, then $g_M \oplus g_N$ defined by

$$(g_M \oplus g_N)_{(p,q)}((X_p, Y_q), (X'_p, Y'_q)) = (g_M)_p(X_p, X'_p) + (g_N)_q(Y_q, Y'_q)$$

is a Riemannian metric on $M \times N$.

**Definition 2.2.** We will call $(M \times N, g_M \oplus g_N)$ the **product Riemannian manifold** of $(M, g_M)$ and $(N, g_N)$.

2. Let $(N, g_N)$ be a Riemannian manifold, and $f : M \to N$ a smooth **immersion**, i.e. $df_p : T_p M \to T_{f(p)} N$ is injective for all $p \in M$. Then the “pull-back metric” $f^* g_N$ on $M$ defined by

$$(f^* g_N)_p(X_p, Y_p) = (g_N)_{f(p)}(df_p(X_p), df_p(Y_p))$$

is a Riemannian metric on $M$.

**Definition 2.3.** We also call $f^* g_N$ the **induced metric** on $M$ (w.r.t. $f$).
(3) Let \((N, g_N)\) be a Riemannian manifold, and \(M \subset N\) be an immersed submanifold. Then the inclusion map \(\iota : M \rightarrow N\) is an immersion, which defines an induced Riemannian metric on \(M\).

**Definition 2.4.** We call \((M, \iota^*g_N)\) a Riemannian submanifold of \((N, g_N)\).

Note that in this case \((\iota^*g_N)_p\) is just the restriction of \(g_N\) onto \(T_pM \subset T_pN\).

(4) Let \((M, g)\) be any Riemannian manifold, and \(u : M \rightarrow \mathbb{R}\) an arbitrary smooth function on \(M\). Then \(e^u g\) defined by
\[
(e^u g)_p(X_p, Y_p) = e^{u(p)} g_p(X_p, Y_p)
\]
is a Riemannian metric on \(M\).

**Definition 2.5.** We say a Riemannian metric \(g'\) is conformal to \(g\) if
\[
g' = e^u g
\]
for some \(u \in C^\infty(M)\).

**Example.** Let \(M = S^2\) be the unit 2-sphere in \(\mathbb{R}^3\). To calculate the induced Riemannian metric, we need to choose a coordinate patch. For example, we can use cylindrical coordinates \(\theta\) and \(z\) to parametrize \(S^2\),
\[
x = \sqrt{1 - z^2} \cos \theta, \quad y = \sqrt{1 - z^2} \sin \theta, \quad z = z,
\]
with \(0 < \theta < 2\pi, -1 < z < 1\). Then
\[
dx = \frac{-z}{\sqrt{1 - z^2}} \cos \theta dz - \sqrt{1 - z^2} \sin \theta d\theta
\]
and
\[
dy = \frac{-z}{\sqrt{1 - z^2}} \sin \theta dz + \sqrt{1 - z^2} \cos \theta d\theta.
\]
It follows
\[
g_{S^2} = [dx \otimes dx + dy \otimes dy + dz \otimes dz]|_{S^2}
\]
\[
= \frac{z^2}{1 - z^2} dz \otimes dz + (1 - z^2) d\theta \otimes d\theta + dz \otimes dz
\]
\[
= \frac{1}{1 - z^2} dz \otimes dz + (1 - z^2) d\theta \otimes d\theta.
\]