LECTURE 1: THE RIEMANNIAN METRIC

0. Course Information

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Instructor: Zuoqin Wang

Email: wangzuoq@ustc.edu.cn

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Lecture time/room: Monday 9:45-11:20 AND Saturday 19:30-21:05 @ 5505

Course webpage: http://staff.ustc.edu.cn/∼wangzuoq/Courses/16S-RiemGeom/

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Course grades = exercises (40%) + midterm exam(30%) + final essay (30%)

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Reference books (not required. **Course notes will be posted online**.)

- *Riemannian geometry* by de Carmo
- *Riemannian geometry* by S. Gallot, D. Hulin and J. Lafontaine.
- *Riemannian geometry* by P. Peterson

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Rough plan of this course:

- Basic Riemannian geometry I: metric, connection and curvature
  - Riemannian metric, distance, length, volume
  - Connections, parallel transport, Levi-Civita
  - Sectional, Ricci and scalar curvature
  - Special classes of Riemannian manifolds
- Basic Riemannian Geometry II: geodesics
  - Exponential map, normal coordinates
  - Jacobi field, variational formulae
  - Index form, Morse index theorem
- Topology of Riemannian manifolds
  - Cartan-Hardamard, Bonnet-Myers, Synge
  - Comparison theorems: Rauch, Toponogov, volume etc
  - Critical point theory of distance function
- Analysis on Riemannian manifolds (if time permits)
  - Weitzenbock, towards Hodge theory
  - Laplace operator: eigenvalues, eigenfunctions

Einstein Summation Convention: If in a term the same index appears twice, both as upper and a lower index, that term is assumed to be summed over all possible values of that index (usually from 1 to the dimension). For example,

$$a_i b^i := \sum_i a_i b^i, \quad a^{ijkl} b_{il} c_j := \sum_{i,j,l} a^{ijkl} b_{il} c_j.$$

1. The Riemannian metric

Let M be a smooth manifold of dimension m. Recall that this implies locally near every point $p \in M$ there is a neighborhood U of p which is diffeomorphic to a domain in $\mathbb{R}^m$. Moreover, if we denote by $\{x^1, \cdots, x^m\}$ the coordinate functions on U, then the tangent space T_pM is spanned by the vectors $\{\partial_1, \cdots, \partial_m\}$, and its dual T_p^*M is spanned by $\{dx^1, \cdots, dx^m\}$.

Definition 1.1. A *Riemannian metric* g on M is an assignment of an inner product $g_p(\cdot,\cdot) = \langle \cdot, \cdot \rangle_p$ on T_pM for each $p \in M$ that depends smoothly on p.

Remarks. (1) "Smooth dependence" $\iff$ if X, Y are two smooth vector fields on an open subset $U \subset M$, then $f(p) = \langle X_p, Y_p \rangle_p$ is a smooth function on U.

(2) g itself is NOT a *metric* (aka a *distance function*) on M. Recall that a distance function on M is a continuous function $d : M \times M \to \mathbb{R}$ so that for all $p, q, r \in M$,

- $d(p,q) \geq 0$, and $d(p,q) = 0$ if and only if $p = q$;
- $d(p,q) = d(q,p)$;
- $d(p,r) \leq d(p,q) + d(q,r)$.

However, we will see soon that g induces a natural distance function d on M.

One can represent the Riemannian metric g using local coordinates as follows. Let $\{U, x^1, \cdots, x^m\}$ be a coordinate patch. We denote

$$g_{ij}(p) = \langle \partial_i, \partial_j \rangle_p.$$

Then for any smooth vector fields $X = X^i \partial_i$ and $Y = Y^j \partial_j$ in U,

$$\langle X_p, Y_p \rangle_p = X^i(p) Y^j(p) \langle \partial_i, \partial_j \rangle_p = g_{ij}(p) X^i(p) Y^j(p).$$

So locally we can write

$$g = g_{ij} dx^i \otimes dx^j.$$

It is easy to see that the coefficients g_{ij} have the following properties:

- For all i, j, $g_{ij}(p)$ is smooth in p.
- $g_{ij} = g_{ji}$, so the matrix $(g_{ij}(p))$ is symmetric at any p.
- The matrix $(g_{ij}(p))$ is also positive definite for any p.

This gives another description of a Riemannian metric g:

A Riemannian metric g is a smooth symmetric $(0,2)$-tensor field that is positive definite everywhere.

Note that each matrix (g_{ij}) is postive definite. We will denote by (g^{ij}) the inverse matrix of (g_{ij}), i.e. they satisfies

$$g_{ij}g^{jk} = \delta_i^k.$$

Then the matrix (g^{ij}) is again positive definte, and we use it to define a *dual* inner product structure on T_p^*M for each p. More explicitly, for any 1-forms $\omega = \omega_i dx^i$ and $\eta = \eta_i dx^i$ on U,

$$\langle \omega, \eta \rangle_p^* := g^{ij}(p)\omega_i(p)\eta_j(p).$$

Since g is non-degenerate and bilinear on T_pM, it gives us an isomorphism between TM and T^*M via

$$\flat : TM \to T^*M, \qquad \flat(X)(Y) := g(X, Y).$$

(Pronunciation of $\flat$: flat)

It is not hard to check that $\flat$ is a vector bundle isomorphism. In local coordinates, if we denote $X = X^i\partial_i$ and take $Y = \partial_j$ for each j, then

$$\flat(X)(\partial_j) = g(X, \partial_j) = g_{ij}X^i,$$

so we conclude

$$\flat(X^i\partial_i) = g_{ij}X^i dx^j.$$

We will denote the inverse map of $\flat$ by

$$\sharp : T^*M \to TM.$$

(Pronunciation of $\sharp$: sharp)

Locally

$$\sharp(w_i dx^i) = g^{ij}w_i\partial_j.$$

Note that $\flat$ is "lowering the indeces", i.e. change the coefficient from X^i to $g_{ij}X^i$, using g_{ij}, while $\sharp$ is "raising the indeces" via g^{ij}. We will call $\flat$ and $\sharp$ the *musical isomorphisms*. [In music, the symbol $\flat$ means lower in pitch while the symbol $\sharp$ means higher in pitch.] Note that they are actually defined pointwise, and the dual inner product on T_p^*M we mentioned above is merely

$$\langle \omega, \eta \rangle_p := g_p(\sharp\omega, \sharp\eta).$$

Remark. More generally, the Riemannian inner product g induced an natural inner product g on $(T_pM)^{\otimes k} \otimes (T_p^*M)^{\otimes l}$: Suppose $e_1, \cdots, e_m$ is an orthonormal basis of T_pM, and $e^1, \cdots, e^m$ its dual basis. Then the induced inner product is defined so that

$$\{e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l}\}$$

form an orthonormal basis.

2. Riemannian manifolds

Let M be a smooth manifold.

Definition 2.1. Let g be Riemannian metric on M. Then we call the pair (M, g) a *Riemannian manifold.*

[Note: Sometimes we omit g and say M is a Riemannian manifold.]

The simplest example of Riemannian manifold is

Example. The standard inner product on $\mathbb{R}^m$ defines a *canonical Riemannian metric* g_0 on $\mathbb{R}^m$ via

$$g_0(X, Y) = \sum_i X^i Y^i.$$

Alternatively, this means the matrix (g_{ij}) is the identity matrix:

$$(g_0)_{ij} = \delta_{ij}.$$

In the notion of tensors, we can write

$$g_0 = dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m.$$

More generally, for any positive definite $m \times m$ matrix $A = (a_{ij})$, the formula

$$g_p^A(X_p, Y_p) := X_p^T A Y_p$$

defines a Riemannian metric on $\mathbb{R}^m$ in which case $g_{ij}^A = a_{ij}$. Equivalently,

$$g^A = \sum_{i,j} a_{ij} dx^i \otimes dx^j.$$

There are many ways to construct new Riemannian manifolds from old, for example,

(1) Let (M, g_M) and (N, g_N) be two Riemannian manifolds, then $g_M \oplus g_N$ defined by

$$(g_M \oplus g_N)_{(p,q)}((X_p, Y_q), (X_p', Y_q')) = (g_M)_p(X_p, X_p') + (g_N)_q(Y_q, Y_q')$$

is a Riemannian metric on $M \times N$.

Definition 2.2. We will call $(M \times N, g_M \oplus g_N)$ the *product Riemannian manifold* of (M, g_M) and (N, g_N).

(2) Let (N, g_N) be a Riemannian manifold, and $f : M \to N$ a smooth *immersion*, i.e. $df_p : T_pM \to T_{f(p)}N$ is injective for all $p \in M$. Then the "pull-back metric" f^*g_N on M defined by

$$(f^*g_N)_p(X_p, Y_p) = (g_N)_{f(p)}(df_p(X_p), df_p(Y_p))$$

is a Riemannian metric on M.

Definition 2.3. We also call f^*g_N the *induced metric* on M (w.r.t. f).

(3) Let (N, g_N) be a Riemannian manifold, and $M \subset N$ be an immersed submanifold. Then the inclusion map $\iota : M \to N$ is an immersion, which defines an induced Riemannian metric on M.

Definition 2.4. We call $(M, \iota^* g_N)$ a *Riemannian submanifold* of (N, g_N).

Note that in this case $(\iota^* g_N)_p$ is just the restriction of g_N onto $T_pM \subset T_pN$.

(4) Let (M, g) be any Riemannian manifold, and $u : M \to \mathbb{R}$ an arbitrary smooth function on M. Then $e^u g$ defined by

$$(e^u g)_p(X_p, Y_p) = e^{u(p)} g_p(X_p, Y_p)$$

is a Riemannian metric on M.

Definition 2.5. We say a Riemannian metric g' is *conformal* to g if

$$g' = e^u g$$

for some $u \in C^\infty(M)$.

Example. Let $M = S^2$ be the unit 2-sphere in $\mathbb{R}^3$. To calculate the induced Riemannian metric, we need to choose a coordinate patch. For example, we can use cylindrical coordinates θ and z to parametrize S^2,

$$x = \sqrt{1-z^2}\cos\theta,\ y = \sqrt{1-z^2}\sin\theta,\ z = z,$$

with $0 < \theta < 2\pi, -1 < z < 1$. Then

$$dx = \frac{-z}{\sqrt{1-z^2}}\cos\theta dz - \sqrt{1-z^2}\sin\theta d\theta$$

and

$$dy = \frac{-z}{\sqrt{1-z^2}}\sin\theta dz + \sqrt{1-z^2}\cos\theta d\theta.$$

It follows

$$\begin{aligned} g_{S^2} &= [dx \otimes dx + dy \otimes dy + dz \otimes dz]|_{S^2} \\ &= \frac{z^2}{1-z^2} dz \otimes dz + (1-z^2) d\theta \otimes d\theta + dz \otimes dz \\ &= \frac{1}{1-z^2} dz \otimes dz + (1-z^2) d\theta \otimes d\theta. \end{aligned}$$