LECTURE 2: THE RIEMANNIAN DISTANCE

1. The existence of Riemannian metrics

We first define the notion of "equivalence" in the Riemannian world.

Definition 1.1. Let (M, g_M) and (N, g_N) be two Riemannian manifolds. A diffeomorphism $\varphi : M \to N$ is called an *isometry* if $g_M = \varphi^* g_N$.

Remark. We call φ a *local isometry* if it is a local diffeomorphism and satisfies $g_M = \varphi^* g_N$. In other words, φ is a local isometry if and only if for any $p \in M$, there exists a neighborhood U of p in M so that $\varphi : U \to \varphi(U)$ is an isometry.

Obviously

- If $\varphi : M \to N$ is an isometry, the $\varphi^{-1} : N \to M$ is an isometry.
- If $\varphi : M \to N$ and $\psi : N \to P$ are two isometries, then the composition
$$\psi \circ \varphi : M \to P$$
is again an isometry.

In particular, if we let
$$\mathrm{Isom}(M, g) = \{\varphi : M \to M \mid \varphi \text{ is an isometry}\}.$$
Then $\mathrm{Isom}(M, g)$ is a group. It is a subgroup of the diffeomorphism group
$$\mathrm{Diff}(M) = \{\varphi : M \to M \mid \varphi \text{ is a diffeomorphism}\}.$$

Definition 1.2. We call $\mathrm{Isom}(M, g)$ the *isometry group* of (M, g).

Remark. For simplicity suppose M is compact. Then one can regard $\mathrm{Diff}(M)$ as a *huge* Lie group, or more precisely, an infinite dimensional Lie group whose Lie algebra is the Lie algebra of all vector field on M. Then the isometry group $\mathrm{Isom}(M, g)$ is much nicer than $\mathrm{Diff}(M)$: it *is* a Lie group (so in particular it is of finite dimensional) whose Lie algebra consists of those vector fields which are called *Killing vector fields*. So the Riemannian structure is much more *rigid* than the smooth structure.

The first remarkable theorem in this course is

Theorem 1.3. *There exist* (many) *Riemannian metrics on any smooth manifold M.*

We shall give two proofs of this theorem.

The first proof. We first take a locally finite covering of M by coordinate patches $\{U_\alpha, x_\alpha^1, \cdots, x_\alpha^m\}$. It is clear that one can choose a Riemannian metric g_α on each U_α, e.g. one may take

$$g_\alpha = \sum dx_\alpha^i \otimes dx_\alpha^i.$$

Let $\{\rho_\alpha\}$ be a partition of unity subordinate the chosen covering $\{U_\alpha\}$. We define

$$g = \sum_\alpha \rho_\alpha g_\alpha.$$

Note that this is in fact a finite sum in the neighborhood of each point. It is positive definite since for any $p \in M$, there always exist some α such that $\rho_\alpha(p) > 0$. So it is a Riemannian metric on M. $\square$

Second proof. According to the famous Whitney embedding theorem, any smooth manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$ as a submanifold. Obviously $\mathbb{R}^{2m+1}$ admits many Riemannian metrics, e.g. any smooth map

$$g : \mathbb{R}^{2m+1} \to \mathrm{PosSym}(2m+1) \subset \mathrm{Sym}(2m+1)$$

defines a Riemannian metric on $\mathbb{R}^{2m+1}$, where $\mathrm{PosSym}(2m+1)$ is the set of all positive definite $(2m+1) \times (2m+1)$ matrices, which is open in $\mathrm{Sym}(2m+1)$. $\square$

Remarks. Let M be any smooth manifold.

(1) Obviously if g_1, g_2 are two Riemannian metrics on M, so is $ag_1 + bg_2$ for $a, b > 0$. In fact the set of Riemannian metrics on any given manifold M form an (infinitely dimensional) positive convex cone.
(2) In the second proof we used the Whitney embedding theorem. For Riemannian manifolds (M, g), there is a much stronger embedding theorem due to the famous Nobel prize winner John Nash,

   Theorem 1.4 (Nash embedding theorem)**.** *Any Riemannian manifold* (M, g) *can be* isometrically embedded *into the standard* $(\mathbb{R}^N, g_0)$ *for* N *large enough. In other words, one can find an embedding* $\varphi : M \to \mathbb{R}^N$ *so that* $\varphi^* g_0 = g_M$*.*

   In fact, one can take $N = \frac{m(3m+11)}{2}$ for M compact and $N = \frac{m(m+1)(3m+11)}{12}$ for M noncompact. The proof is much harder than Whitney's theorem.
(3) We have mentioned that $\mathrm{Diff}(M)$ is infinitely dimensional while $\mathrm{Isom}(M, g)$ is finite dimensional (at least for compact M). So for *most* diffeomorphism $\varphi : M \to M$, the pull back $\varphi^* g$ is a Riemannian metric on M that is *different* from g.
(4) Of course a natural question is: Given a manifold, can one find a Riemannian metric that is "best" in some sense? This is one of the main target in Riemannian geometry. In this course we shall define various kind of invariants of Riemannian metrics, and we shall study the relations between these invariants and the topology of the underlying manifold.

2. The Riemannian distance

To define the Riemannian distance between two points, we first need to define the length of a curve. Let $\gamma : [a, b] \to M$ be a smooth immersed *parametric* curve in M. Then for any $t \in [a, b]$, $\dot{\gamma}(t) = d\gamma(\frac{d}{dt})$ is a tangent vector in $T_{\gamma(t)}M$. We shall always assume that the parametrization is*regular*, i.e. $\dot{\gamma}(t) \neq 0$ for all t. As in undergraduate differential geometry course, we define

Definition 2.1. The *length* of γ is

$$\mathrm{Length}(\gamma) = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} dt$$

Lemma 2.2. Length(γ) *is independent of the choices of* regular *parametrizations.*

Proof. Let $\gamma_1 : [c, d] \to M$ be another parametrization of the same *geometric curve* as γ. Then there exists a smooth function $t_1 = t_1(t)$ maps $[a, b]$ to $[c, d]$ so that $\gamma_1(t_1(t)) = \gamma(t)$. Moreover, the function $t_1 = t_1(t)$ is either strictly increasing, or strictly decreasing, since both parametrizations are regular. Now the conclusion follows from the standard change of variable formula. □

So the length of the *geometric curve* γ is well-defined. A change of variable argument also yields

Lemma 2.3. *Let $\varphi : M \to N$ be an isometry, and γ be a curve in M. Then*

$$\mathrm{Length}_M(\gamma) = \mathrm{Length}_N(\varphi(\gamma)).$$

Now let $\gamma : [a, b] \to M$ be a regular curve. We will call

$$s(t) = \int_a^t \sqrt{\langle \dot{\gamma}(\tau), \dot{\gamma}(\tau) \rangle_{\gamma(\tau)}} d\tau$$

the *arc-length function* of γ. Obviously $s = s(t)$ is a strictly increasing function mapping the interval $[a, b]$ to the interval $[0, \mathrm{Length}(\gamma)]$. We will denote by $t = t(s)$ the inverse function of $s = s(t)$.

One can reparametrize γ using the arc-length parameter s,

$$\gamma_1(s) = \gamma(t(s)), \qquad 0 \leq s \leq \mathrm{Length}(\gamma).$$

This is called the *arc-length parametrization.* It has the following nice property

Proposition 2.4. *For the arc-length parametrization,* $\langle \dot{\gamma}_1(s), \dot{\gamma}_1(s) \rangle_{\gamma_1(s)} \equiv 1$.

Proof. At $s = s(t)$, we have $t'(s) = (s'(t))^{-1} = \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle^{-1/2}$. So

$$\langle \dot{\gamma}_1(s), \dot{\gamma}_1(s) \rangle_{\gamma_1(s)} = \langle \dot{\gamma}(t)t'(s), \dot{\gamma}(t)t'(s) \rangle_{\gamma_1(s(t))} = t'(s)^2 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)} = 1.$$

□

Remark. Discussions above can be easily extended to *piecewise smooth curves* in M.

Now we are ready define a metric (i.e. a distance function) on a Riemannian manifold (M, g). For any $p, q \in M$, let

$$\mathcal{C}_{pq} = \{\gamma : [a, b] \to M \mid \gamma \text{ is piecewise smooth and } \gamma(a) = p, \gamma(b) = q\}.$$

This set is nonempty since M is (always assumed to be) connected.

Definition 2.5. The *distance* between p and q on (M, g) is

$$\mathrm{dist}(p, q) = \inf\{\mathrm{Length}(\gamma) \mid \gamma \in \mathcal{C}_{pq}\}.$$

We first note that if $M = \mathbb{R}^m$ and $g = g_0$ the standard metric, then dist is *the distance function* on $\mathbb{R}^m$. For a general Riemannian metric, it is easy to check that the function

$$\mathrm{dist} : M \times M \to \mathbb{R}$$

satisfis *most* conditions of a distance function, i.e. for any $p, q, r \in M$,

(a) $\mathrm{dist}(p, p) = 0, \quad \mathrm{dist}(p, q) \geq 0$;
(b) $\mathrm{dist}(p, q) = \mathrm{dist}(q, p)$;
(c) $\mathrm{dist}(p, r) \leq \mathrm{dist}(p, q) + \mathrm{dist}(q, r)$.

Now let's prove

Theorem 2.6. *The distance function* dist *makes M into a metric space.*

Proof. It remains to show that for $p \neq q$, we must have

$$\mathrm{dist}(p, q) > 0.$$

We take a chart $\{\varphi, U, V\}$ around q with

$$\varphi(q) = 0 \in V = B_1(0) \subset \mathbb{R}^m,$$

so that $p \notin U$. We let λ be the smallest eigenvalue of

$$h = (\varphi^{-1})^* g_U$$

at all points in $\overline{B_{1/2}(0)}$. Let γ be an arbitray curve starting from $0 = \varphi(q)$ and ending at some point on $\partial B_{1/2}(0)$, and $\tilde{\gamma}$ be the first piece of curve γ that sits totally in $B_{1/2}(0)$ (so $\tilde{\gamma}$ starts at q and still ends at some point on $\partial B_{1/2}(0)$), reparametrized with parameters in $[0, 1]$. Then

$$\begin{aligned}\mathrm{Length}_h(\gamma) \geq \mathrm{Length}_h(\tilde{\gamma}) &= \int_0^1 \sqrt{\langle \dot{\tilde{\gamma}}, \dot{\tilde{\gamma}} \rangle_h}\, dt \\ &\geq \sqrt{\lambda} \int_0^1 \sqrt{\langle \dot{\tilde{\gamma}}, \dot{\tilde{\gamma}} \rangle_{g_0}}\, dt = \sqrt{\lambda}\, \mathrm{Length}_{g_0}(\tilde{\gamma}) \geq \frac{\sqrt{\lambda}}{2}.\end{aligned}$$

Since any curve from p to q must intersect $\varphi(\partial B_{1/2}(0))$ at some point, we conclude that

$$\mathrm{dist}(p, q) \geq \frac{\sqrt{\lambda}}{2} > 0,$$

as desired. $\square$

Moreover,

Proposition 2.7. *For any fixed p, the function*

$$f(\cdot) = \mathrm{dist}(\cdot, p)$$

is continuous on M.

Proof. Let q_i be a sequnce of points that tends to q, i.e. for any k, there exists $N(k)$ such that for all $i \ge N(k)$, $\varphi(q_i) \in B_{1/k}(0)$. We want to prove $f(q_i) \to f(q)$. By triangle inequality (see (c) above), we have

$$|f(q_i) - f(q)| \le \mathrm{dist}(q, q_i).$$

So it suffices to prove $\mathrm{dist}(q, q_i) \to 0$ as $i \to \infty$.

To proceed, as in the proof of the previous theorem we take a coordinate patch $\{\varphi, U, V\}$ centered at q with $\varphi(U) = V = B_1(0) \subset \mathbb{R}^m$. Let $h = (\varphi^{-1})^* g|_U$ be the induced metric on V. Let Λ be the greatest eigenvalue of h at all points in $\overline{B_{1/2}(0)}$. So if we take

$$\tilde{\gamma}_i : [0,1] \to V, \quad \tilde{\gamma}_i(t) = t\varphi(q_i)$$

be the "straight line segment" from $0 = \varphi(q)$ to $\varphi(q_i)$, then for $i \ge N(k)$, (with g_0 = the canonical metric on $V \subset \mathbb{R}^m$)

$$\mathrm{Length}_h(\tilde{\gamma}_i) = \int_0^1 \sqrt{\langle \dot{\tilde{\gamma}}_i, \dot{\tilde{\gamma}}_i \rangle_h} dt \le \sqrt{\Lambda} \int_0^1 \sqrt{\langle \dot{\tilde{\gamma}}_i, \dot{\tilde{\gamma}}_i \rangle_{g_0}} dt = \sqrt{\Lambda}\, \mathrm{Length}_{g_0}(\tilde{\gamma}_i) \le \frac{\sqrt{\Lambda}}{k}.$$

Since (U, g) is isometric to (V, h), we conclude that

$$\mathrm{Length}_g(\gamma_i) = \mathrm{Length}_h(\tilde{\gamma}_i) \le \frac{\sqrt{\Lambda}}{k},$$

where $\gamma_i = \varphi^{-1}(\tilde{\gamma}_i)$ is a curve from q to q_i. It follows that $\mathrm{dist}(q, q_i) \le \frac{\sqrt{\Lambda}}{k}$ for all $i \ge N(k)$. This completes the proof. $\square$

As a consequence, we have

Theorem 2.8. *The metric topology on M induced by the metric* dist *coincides with the manifold topology on M.*

Proof. From the proof of theorem 2.6, we have already seen that any "manifold open set" contains a metric open ball. Conversely from the continuity of $f(\cdot) = \mathrm{dist}(p, \cdot)$ we know that any metric open ball is also open in the manifold topology. So the two topologies coincide. $\square$

Remark. Note the triangle inequality implies that the "distance to p" function

$$f(q) = \mathrm{dist}(p, q)$$

is not only continuous, but also absolutely continuous. In fact, as we will see later, this function is smooth in a "punctured neighborhood" of p. (It is not smooth at p of course. It also might have singularities at other points.)