LECTURE 3: THE RIEMANNIAN MEASURE

1. The Riemannian measure

Let (M, g) be a Riemannian manifold, and K a compact subset in some coordinate patch $(U, x^1, \cdots, x^m)$ such that $x(K)$ is measurable. We define the *volume* of K to be

$$\mathrm{Vol}(K) := \int_{x(K)} \sqrt{G \circ x^{-1}}\, dx^1 \cdots dx^m,$$

where $G = \det(g_{ij})$, $g_{ij} = g(\partial_i, \partial_i)$, and $dx^1 \cdots dx^m$ the Lebesgue measure on $\mathbb{R}^m$.

Lemma 1.1. *The definition above is independent of the choice of coordinate patch.*

Proof. Let $\{\tilde{U}, y^1, \cdots, y^m\}$ be another coordinate patch containing K, then

$$\partial_i^x|_p = J_i^k(x(p))\partial_k^y|_p,$$

where $J_i^k = \frac{\partial(y^k \circ x^{-1})}{\partial x^i}$ is the Jacobian element of $y \circ x^{-1} : x(U) \to y(\tilde{U})$ (diffeomorphism between open sets in $\mathbb{R}^m$). It follows that

$$(g_{ij}^x) = J^T(g_{kl}^y)J.$$

As a consequence, we get

$$\sqrt{G^x(p)} = \sqrt{G^y(p)}\ |\det(J(x(p)))|$$

and thus

$$\sqrt{G^y \circ y^{-1}}dy^1 \cdots dy^m = \sqrt{G^y \circ y^{-1}(y \circ x^{-1})}\ |\det(J)| dx^1 \cdots dx^m = \sqrt{G^x \circ x^{-1}} dx^1 \cdots dx^m,$$

where the first equality follows from the change of variables in $\mathbb{R}^m$. □

Obviously the definition above extends to the volume of any reasonable ("measurable") subset of K. In general, to define the volume of a reasonably nice ("measurable") set A that need not be in one coordinate patch, we use the partition of unity argument. More precisely, we pick a local finite atlas $\{U_\alpha, x_\alpha^1, \cdots, x_\alpha^m\}$ of M and a partition of unity $\{\rho_\alpha\}$ subordinate to this atlas. Now we can set

$$\mathrm{Vol}(A) = \sum_\alpha \int_{x^\alpha(A \cap U_\alpha)} (\rho_\alpha \sqrt{G^\alpha}) \circ (x^\alpha)^{-1} dx_\alpha^1 \cdots dx_\alpha^m,$$

as long as each integral in the sum exists. This leads to

Definition 1.2. The *Riemannian volume element* (or *volume density*) on (M, g) is

$$\boxed{d\mathrm{Vol} = \sum_\alpha (\rho_\alpha \sqrt{G^\alpha}) \circ (x^\alpha)^{-1} dx_\alpha^1 \cdots dx_\alpha^m.}$$

Remarks.

1. It is standard to check that this definition is independent of the choices of the atlas and is independent of the choices of the partition of unity.
2. In the case the sum diverges, we say that the volume of A is infinite.
3. In the above definition, we don't assume M to be oriented or compact. If M is oriented and the coordinate system is taken to be orientation-preserving, then the volume density is actually a *positive n-form*,

$$\omega_g = \sqrt{G}dx^1 \wedge \cdots \wedge dx^m,$$

and we call it the *Riemannian volume form.*

Now with the Riemannian volume element dVol we can integrate functions on (M, g). Let $C_c^0(M)$ be the space of compactly supported continuous functions on M. Then for any $f \in C_c^0(M)$, we can define

$$\int_M f d\mathrm{Vol} = \sum_\alpha \int_{U_\alpha} f \circ (x^\alpha)^{-1} (\rho_\alpha \sqrt{G^\alpha}) \circ (x^\alpha)^{-1} dx_\alpha^1 \cdots dx_\alpha^m.$$

This integral is well defined, and satisfies all the properties that the usual Lebesgue integral should satisfy. As usual, for any $1 \le p < \infty$ one can define the L^p norm on C_c^∞ via

$$\|f\|_{L^p} := \left(\int_M |f|^p d\mathrm{Vol} \right)^{1/p}.$$

The completion of C_c^∞ under the L^p norm is called $L^p(M)$. Similarly one can define $L^\infty(M)$.

In the special case $p = 2$, one can define an inner product structure on $L^2(M)$,

$$\langle f_1, f_2 \rangle_{L^2} := \int_M f_1 \bar{f}_2 d\mathrm{Vol}$$

which make $L^2(M)$ into a Hilbert space.

2. The gradient and divergence

Let (M, g) be a Riemannian manifold. Recall that the musical isomorphism $\sharp : T^*M \to TM$ maps a 1-form to a vector field. Locally it is given by

$$\sharp(w_i dx^i) = g^{ij} w_i \partial_j,$$

It is also characterized by the relation that for any 1-form ω and vector field X,

$$g(\sharp\omega, X) = \omega(X).$$

Now suppose f is a smooth function on M. Then df is a 1-form on M.

Definition 2.1. The *gradient vector field* of f is $\nabla f = \sharp(df)$.

Note that the definition is equivalent to say that for any vector field $X = X^i\partial_i$,

$$g(\nabla f, X) = Xf = X^i\partial_i f.$$

In local coordinates, we have

$$\boxed{\nabla f = g^{ij}\partial_i f \partial_j.}$$

In particular, for $g = g_0$ in $\mathbb{R}^m$, we get the ordinary gradient of f.

As in multivariable calculus, the gradient vector field of a function is always perpendicular to its level sets:

Lemma 2.2. *Suppose f is a smooth function on M and c is a regular value of f. Then the gradient vector field ∇f is* perpendicular *to the level set $f^{-1}(c)$.*

Proof. Since c is a regular value, then $f^{-1}(c)$ is a submanifold of M. Let X be a vector field tangent to $f^{-1}(c)$. Then we learned from manifold theory that $Xf = 0$ on $f^{-1}(c)$. It follows

$$g(\nabla f, X) = Xf = 0$$

on $f^{-1}(c)$. So ∇f is perpendicular to $f^{-1}(c)$. □

Now suppose X is a smooth vector field on M. Take a coordinate patch $(U, x^1, \cdots, x^m)$ on M, then the volume element

$$\omega_g = \sqrt{G}dx^1 \wedge \cdots \wedge dx^m$$

is locally an n-form on U, and two different choices of charges may induce ω_g's that differ by a negative sign. We define

Definition 2.3. The *divergence* of X is the function $\mathrm{div}(X)$ on M such that

$$(\mathrm{div}X)\omega_g = d\{\iota(X)\omega_g\}.$$

Remark. We don't require M to be orientable, since for any two choices of charts, ω_g's are either the same or differed by a negative sign, so that $\mathrm{div}(X)$ are still the same. In the case M is orientable, ω_g is a global positive n-form on M. According to the Cartan's magic formula, the definition above is equivalent to

$$\mathcal{L}_X(\omega_g) = \mathrm{div}(X)\omega_g,$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X. This coincides with the geometric definition of divergence in the case of $\mathbb{R}^m$: the divergence of a vector field is the infinitesimal rate of change of the volume element along the vector field.

Now let's calculate $\mathrm{div}(X)$ locally. Let $X = X^i\partial_i$, then

$$\begin{aligned}(\mathrm{div}X)\sqrt{G}dx^1 \wedge \cdots \wedge dx^m &= d\{\iota(X^i\partial_i)\sqrt{G}dx^1 \wedge \cdots \wedge dx^m\} \\ &= d\{X^i\sqrt{G}(-1)^{i-1}dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m\} \\ &= \partial_i(X^i\sqrt{G})dx^1 \wedge \cdots \wedge dx^m,\end{aligned}$$

so we conclude

$$\boxed{\operatorname{div}(X^i\partial_i) = \frac{1}{\sqrt{G}}\partial_i(X^i\sqrt{G}).}$$

As a consequence, we see

$$\operatorname{div}(fX) = f\operatorname{div}X + (\partial_i f)X^i = f\operatorname{div}X + g(\nabla f, X).$$

Theorem 2.4 (The Divergence theorem I)**.** *Let X be a smooth vector field with compact support on a Riemannian manifold (M,g), then*

$$\int_M \operatorname{div}(X)d\mathrm{Vol} = 0.$$

Proof. Without loss of generality, we assume that X is supported in a local chart $(U, x^1, \cdots, x^m)$ and thus $X = X^i\partial_i$ with $X^i \in C_c^\infty(U)$. Then

$$\begin{aligned}\int_M \operatorname{div}(X)d\mathrm{Vol} &= \int_U \frac{1}{\sqrt{G}}\partial_i(X^i\sqrt{G})d\mathrm{Vol}\\ &= \int_{x(U)} \partial_i(X^i\sqrt{G}\circ x^{-1})dx^1\cdots dx^m = 0.\end{aligned}$$

□

3. The Laplacian

Let (M,g) be a Rimannian manifold.

Definition 3.1. For any smooth function f, we define the *Laplacian* of f to be

$$\Delta f = -\operatorname{div}(\nabla f).$$

Locally, Δf is given by

$$\Delta f = -\operatorname{div}(g^{ij}\partial_i f\partial_j) = -\frac{1}{\sqrt{G}}\partial_i(\sqrt{G}g^{ij}\partial_j f),$$

i.e.

$$\boxed{\Delta = -\frac{1}{\sqrt{G}}\partial_i(\sqrt{G}g^{ij}\partial_j).}$$

We shall call Δ the *Laplace-Beltrami* operator.

Theorem 3.2 (Green's formula I)**.** *Suppose f and h are smooth function on M and either f or h is compactly supported. Then*

$$\int_M f\Delta h\ d\mathrm{Vol} = \int_M g(\nabla f, \nabla h)d\mathrm{Vol} = \int_M h\Delta f\ d\mathrm{Vol}.$$

Proof. We have seen

$$\mathrm{div}(fX) = f\mathrm{div}X + g(\nabla f, X).$$

It follows

$$\mathrm{div}(f\nabla h) = -f\Delta h + g(\nabla f, \nabla h).$$

Now the theorem follows from the fact that $f\nabla h$ is compactly supported. □

In particular if M is compact, then any function is compactly supported. Replacing h by $\overline{h}$, we can rewrite the above formula as

$$\langle f, \Delta h\rangle = \langle \Delta f, h\rangle.$$

In other words, we get

Corollary 3.3. *If M is compact, then Δ is symmetric on $L^2(M)$.*

As another immediate consequence, we see that Δ is a positive operator:

Corollary 3.4. *If M is compact, then $\langle \Delta f, f\rangle \geq 0$.*

Remark. Both the divergence theorem and the Green's formula can be generalized to the case where M is a *compact Riemannian manifold with boundary*, i.e. M is

- an m dimensional smooth manifold with boundary
- M is also a compact subset of an m dimensional Riemannian manifold N
- The Riemannian structure on M coincide with that of N

So ∂M carries

(1) an outward normal vector field ν
(2) an induced Riemannian metric from g_N, and thus a volume density dA.

Then for any smooth vector field X on M and any smooth functions f, h on M,

•(Divergence Theorem II) $\displaystyle\int_M \mathrm{div}(X)d\mathrm{Vol} = \int_{\partial M} g(X,\nu)dA,$

•(Green's formula II) $\displaystyle\int_M f\Delta h\ d\mathrm{Vol} = -\int_M g(\nabla f, \nabla h)\ d\mathrm{Vol} + \int_{\partial M} g(\nu, \nabla h) f\ dA.$