LECTURE 7: DECOMPOSITION OF THE RIEMANN CURVATURE TENSOR

1. Some tensor algebra

Let $V$ be any vector space. Recall that $\wedge^2 V \subset \otimes^2 V$ represents the space of anti-symmetric 2-tensors on $V$, while $S^2 V \subset \otimes^2 V$ represents the space of symmetric 2-tensors on $V$. Any 2-tensor can be decomposed uniquely as the summation of a symmetric 2-tensor and an anti-symmetric 2-tensor. If $\dim V = m$, then

$$\dim \wedge^2 V = \frac{m(m-1)}{2}$$

and

$$\dim S^2 V = \frac{m(m+1)}{2}.$$ 

As a consequence, $S^2(\wedge^2 V^*)$ contains 4-tensors that is symmetric with respect to $(1,2) \leftrightarrow (3,4)$ and is anti-symmetric with respect to $1 \leftrightarrow 2$ and with respect to $3 \leftrightarrow 4$, i.e.


It is a vector space of dimension

(1.1) $$\dim S^2(\wedge^2 V^*) = \frac{m(m-1)(m^2 - m + 2)}{8}.$$ 

We note that the space of 4-forms, $\wedge^4 V^*$, is a subspace of $S^2(\wedge^2 V^*)$ with dimension

(1.2) $$\dim \wedge^4 V^* = \binom{m}{4}.$$ 

Another way to describe the space $S^2(\wedge^2 V^*)$ is the following: Let $\alpha, \beta \in \wedge^2 V^*$ be any two 2-forms, both viewed as skew-symmetric 2-tensors. Recall that the symmetric product of $\alpha$ and $\beta$ is a 4-tensor given by

(1.3) $$(\alpha \odot \beta)(X,Y,Z,W) = \alpha(X,Y)\beta(Z,W) + \alpha(Z,W)\beta(X,Y).$$

Using a basis one has

$$(\alpha \odot \beta)_{ijkl} = \alpha_{ij}\beta_{kl} + \alpha_{kl}\beta_{ij}.$$ 

Obviously $\alpha \odot \beta$ is in $S^2(\wedge^2 V^*)$. Conversely, by dimension counting it is not hard to see that any element in $S^2(\wedge^2 V^*)$ can be written as a linear combination of elements of the form $\alpha \odot \beta$. 

1
For any $T \in S^2(\wedge^2 V^*)$, we define its **Bianchi symmetrization** to be the 4-tensor
\[
T(X, Y, Z, W) = \frac{1}{3}(T(X, Y, Z, W) + T(Y, Z, X, W) + T(Z, X, Y, W)).
\]

**Lemma 1.1.** $b(\alpha \circ \beta) = \frac{1}{3} \alpha \wedge \beta$.

**Proof.** We do the calculation using a basis:
\[
(b(\alpha \circ \beta))_{ijkl} = \frac{1}{3} \left( \alpha_{ij} \beta_{kl} + \alpha_{kl} \beta_{ij} + \alpha_{jk} \beta_{il} + \alpha_{il} \beta_{jk} + \alpha_{ki} \beta_{jl} + \alpha_{jl} \beta_{ki} \right)
\]
\[
= \frac{1}{3} \left( \alpha_{ij} \beta_{kl} - \alpha_{ik} \beta_{jl} + \alpha_{il} \beta_{jk} + \alpha_{jk} \beta_{il} - \alpha_{jl} \beta_{ik} + \alpha_{kl} \beta_{ij} \right)
\]
\[
= \frac{1}{3} (\alpha \wedge \beta)_{ijkl}.
\]
\[ \square \]

As a consequence, we see that the Bianchi symmetrization map
\[
b : S^2(\wedge^2 V^*) \to S^2(\wedge^2 V^*)
\]
has image
\[
\text{Im}(b) = \wedge^4 V^*.
\]

Moreover, by definition it is easy to check that $b$ is a projection, i.e.
\[
b^2 = b.
\]

So from the standard linear algebra one has a direct sum decomposition
\[
S^2(\wedge^2 V^*) = \text{Ker}(b) \oplus \text{Im}(b) = \text{Ker}(b) \oplus \wedge^4 V^*.
\]

Explicitly for any $T \in S^2(\wedge^2 V^*)$, the decomposition is
\[
T = [T - b(T)] + b(T).
\]

We shall denote
\[
\mathcal{C} = \text{Ker}(b).
\]

It is a vector space of dimension
\[
\dim \mathcal{C} = \frac{m(m-1)(m^2 - m + 2)}{8} - \left( \frac{m}{4} \right) = \frac{1}{12} m^2 (m^2 - 1)
\]

The elements in $\mathcal{C}$ are called **curvature-like** tensors.

**Example.** Let $T_1, T_2 \in S^2 V^*$. Consider their **Kulkarni-Nomizu product**
\[
T_1 \heartsuit T_2(X, Y, Z, W) = T_1(X, Z)T_2(Y, W) - T_1(Y, Z)T_2(X, W) - T_1(X, W)T_2(Y, Z) + T_1(Y, W)T_2(X, Z).
\]

It is a 4-tensor in $S^2(\wedge^2 V^*)$. It is easy to see $b(T_1 \heartsuit T_2) = 0$, i.e.
\[
T_1 \heartsuit T_2 \in \mathcal{C}.
\]

So they are all curvature-like tensors.
Now suppose the vector space $V$ is endowed with an inner product $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$, so that one can identify $V^*$ with $V$ using the musical isomorphisms $\flat$ and $\sharp$. In particular, for any element $T$ in $\mathcal{C}$ and for any vectors $X, Y, Z$, the linear map

$$T(Z, X, \cdot, Y) : V \to \mathbb{R},$$

ccan be viewed as an element in $V^*$ and thus be identified with a vector $\sharp T(Z, X, \cdot, Y)$ in $V$. Fixing $X, Y$ but letting $Z$ varies, one gets a linear map $Z \mapsto \sharp T(Z, X, \cdot, Y)$.

**Definition 1.2.** For any curvature like tensor $T \in \mathcal{C}$, the Ricci contraction $c(T)$ of $T$ is the following 2-tensor:

$$c(T)(X, Y) := \text{Tr}(Z \mapsto \sharp T(Z, X, \cdot, Y)).$$

**Lemma 1.3.** $c(T)(X, Y) = c(T)(Y, X)$.

**Proof.** Fix $X$ and $Y$. Let $K : V \to V$ be the map

$$K(Z) = \sharp T(Z, X, \cdot, Y)$$

and let $\tilde{K} : V \to V$ be the map

$$\tilde{K}(Z) = \sharp T(\cdot, X, Z, Y)$$

Then for any $Z, W \in V$, one has

$$\langle K(Z), W \rangle = \langle \sharp T(Z, X, \cdot, Y), W \rangle = T(Z, X, W, Y)$$

$$\langle Z, \tilde{K}(W) \rangle = \langle Z, \sharp T(\cdot, X, W, Y) \rangle = T(Z, X, W, Y).$$

So $\tilde{K}$ is the transpose of $K$. In particular, they have the same trace. But by definition, the trace of $K$ is $c(T)(X, Y)$, while the trace of $\tilde{K}$ is $c(T)(Y, X)$ (since $T$ is in $\mathcal{C} \subset S^2(\wedge^2 V^*)$).

Another way to see the symmetry of $c(T)$ is to write down everything in a basis. If $v^1, \cdots, v^m$ be a basis of $V^*$, and we denote $g^{pq} = g(v^p, v^q)$, then one has [exercise!]

$$c(T)_{ij} = g^{pq} T_{ipjq}$$

Back to the Kulkarni-Nomizu product. Since $g$ is a symmetric 2-tensor, one can define a map

$$\Psi : S^2 V^* \to \mathcal{C}, \quad T \mapsto T \otimes g.$$

As we explained in lecture 1, a metric $g$ on $V$ induces a metric on space of tensors. For example, if $e^1, \cdots, e^m$ is an orthonormal basis of $V^*$ under the metric $g$, then one defines the induced metric on $\otimes^4 V^*$ by requiring that $\{e^{i_1} \otimes e^{i_2} \otimes e^{i_3} \otimes e^{i_4}\}$ be an orthonormal basis. Under the induced metric, one can show that $\Psi$ is “almost” the adjoint of the Ricci contraction map $c$:

**Lemma 1.4.** For any $S \in \mathcal{C}$ and any $T \in S^2 V^*$, one has

$$\langle S, \Psi(T) \rangle = 4 \langle c(S), T \rangle.$$
Proof. We calculate using the orthonormal basis $e^1, \cdots, e^m$:
\[
\langle S, T \wedge g \rangle = S_{ijkl}(T_{ik}g_{jl} - T_{jk}g_{il} - T_{il}g_{jk} + T_{jl}g_{ik}) \\
= S_{ijkl}T_{ik} - S_{iklj}T_{jk} - S_{ikjl}T_{il} + S_{ijlk}T_{jl} \\
= 4S_{ijkl}T_{ik} \\
= 4(c(S))_{ik}T_{ik} = 4\langle c(S), T \rangle.
\]
□

One can repeat the procedure above to define the trace of a symmetric 2-tensor using the metric: If $T$ is any symmetric 2-tensor, then one can regard it as a map
\[
T(X, \cdot) : V \to \mathbb{R}
\]
which can be identified with an element in $V^*$, and thus a vector $\sharp T(X, \cdot)$ in $V$. Again one gets a linear map
\[
X \mapsto \sharp T(X, \cdot).
\]
The trace of this map is a constant now.

**Definition 1.5.** The trace of $T \in S^2V^*$, Tr$(T)$, is the scalar
\[
(1.13) \quad \text{Tr}(T) := \text{trace}(X \mapsto \sharp T(X, \cdot)).
\]

In a basis one has
\[
(1.14) \quad \text{Tr}(T) = g^{pq}T_{pq}.
\]
Note that by using the inner product, one can also write this as
\[
(1.15) \quad \text{Tr}(T) = \langle T, g \rangle,
\]
since the right hand side equals
\[
\langle T, g \rangle = T_{ij}g_{kl}g^{ik}g^{jl} = T_{ij}\delta^i_jg^{ij} = T_{ij}g^{ij}.
\]

**Lemma 1.6.** For any symmetric 2-tensor $T \in S^2V^*$,
\[
(1.16) \quad c(\Psi(T)) = (m - 2)T + \text{Tr}(T)g.
\]

Proof. Again we prove this using the orthonormal basis above. We have
\[
c(T \wedge g)_{ij} = g^{pq}(T_{ij}g_{pq} - T_{pj}g_{iq} - T_{iq}g_{pj} + T_{pq}g_{ij}) \\
= mT_{ij} - T_{ij} - T_{ij} + \text{Tr}(T)g_{ij}.
\]

□

As a result, we can prove

**Proposition 1.7.** The map $\Psi$ is injective for $m > 2$. 


Proof. Suppose $\Psi(T) = 0$. Then
\[
0 = \langle \Psi(T), \Psi(T) \rangle = 4\langle c(\Psi(T)), T \rangle \\
= 4\langle (m - 2)T + \text{Tr}(T)g, T \rangle \\
= 4(m - 2)|T|^2 + 4(\text{Tr}(T))^2.
\]
It follows that $T = 0$ if $m > 2$. \qed

In particular,

Corollary 1.8. $\Psi$ is bijective if $m = 3$.

Proof. Since in this case, both $S^2V^*$ and $\mathcal{C}$ are of 6 dimensional. \qed

2. DECOMPOSITION OF THE RIEMANN CURVATURE TENSOR

Now suppose $(M, g)$ be a Riemannian manifold. As usual, we shall use $S^2(\wedge^2T^*M)$ to represent the sub-bundle of $\otimes^4T^*M$ whose fiber at $p$ is $S^2(\wedge^2T^*_pM)$. By the formulae above,
\[
\dim S^2(\wedge^2T^*_pM) = \frac{m(m - 1)(m^2 - m + 2)}{8},
\]
where $m$ is the dimension of $M$. By this definition, a section $T \in \Gamma(S^2(\wedge^2T^*M))$ is a $(0, 4)$-tensor that satisfies
\[
\]

Example. Last time we showed $Rm \in \Gamma(S^2(\wedge^2T^*M))$.

All the constructions in the previous section works for tensor fields without any difficulty. For example, the Bianchi symmetrization of $T \in \Gamma(S^2(\wedge^2T^*M))$ is
\[
bT(X, Y, Z, W) = \frac{1}{3}(T(X, Y, Z, W) + T(Y, Z, X, W) + T(Z, X, Y, W)).
\]
So $b$ is now a map $b : \Gamma(S^2(\wedge^2T^*M)) \to \Gamma(S^2(\wedge^2T^*M))$. According to the first Bianchi identity, $b(Rm) = 0$.

Definition 2.1. A tensor field $T \in \Gamma(S^2(\wedge^2T^*M))$ is called curvature-like if $T \in \text{Ker}(b)$.

Definition 2.2. We will call the symmetric 2-tensor field
\[
(2.1) \quad \text{Ric} := c(Rm)
\]
the Ricci curvature tensor of $(M, g)$.

Definition 2.3. We will call the function
\[
(2.2) \quad S := \text{Tr}(\text{Ric})
\]
the scalar curvature of $(M, g)$.
Definition 2.4. We will call the symmetric 2-tensor field
\[(2.3)\]
\[A := \frac{1}{m-2} \left( \text{Ric} - \frac{S}{2(m-1)} g \right)\]
the Schouten tensor of \((M, g)\).

Definition 2.5. We call the curvature-like tensor
\[(2.4)\]
\[W = Rm - A \wedge g\]
the Weyl curvature tensor of \((M, g)\).

Lemma 2.6. \(c(W) = 0\).

Proof. Since \(\text{Tr}(g) = \langle g, g \rangle = m\), we have
\[
\text{Tr}(A) = \frac{1}{m-2} \left( S - \frac{S}{2(m-1)} m \right) = \frac{S}{2(m-1)}.
\]
So
\[
c(W) = \text{Ric} - c(A \wedge g) = \text{Ric} - (m-2)A - \text{Tr}(A)g
\]
\[= \text{Ric} - \left( \text{Ric} - \frac{S}{2(m-1)} g \right) - \frac{S}{2(m-1)} g
\]
\[= 0.\]

\[\square\]

Corollary 2.7. If \(m = 3\), then \(W = 0\).

Proof. Since \(m = 3\), there exists symmetric 2-tensor field \(T\) so that \(W = \Psi(T)\). So by lemma 1.4,
\[
\langle W, W \rangle = \langle W, \Psi(T) \rangle = 4\langle c(W), T \rangle = 0.
\]
So \(W = 0\).

\[\square\]

Let’s play with the decomposition
\[(2.5)\]
\[Rm = W + A \wedge g\]
a bit more.

Definition 2.8. We will call
\[(2.6)\]
\[E = \text{Ric} - \frac{S}{m} g\]
the traceless Ricci tensor.

It is called traceless because
\[
\text{Tr}E = \text{Tr}(\text{Ric}) - \frac{S}{m} \text{Tr}(g) = S - \frac{S}{m} m = 0.
\]
Using $E$ we can rewrite the equation (2.5) as

\[(2.7)\quad Rm = W + \frac{1}{m-2}E\otimes g + \frac{S}{2m(m-1)}g\otimes g\]

**Proposition 2.9.** The decomposition (2.7) is an orthogonal decomposition.

**Proof.** We have

$$\langle W, E\otimes g \rangle = \langle W, \Psi(E) \rangle = 4\langle c(W), E \rangle = 0.$$  
Similarly one can prove $\langle W, g\otimes g \rangle = 0$. Finally  
$$\langle E\otimes g, g\otimes g \rangle = \langle E, c(g\otimes g) \rangle = \langle E, (2m-2)g \rangle = (2m-2)\text{Tr}(E) = 0.$$  

□

Our final goal for today is to prove

**Theorem 2.10.** We have

\[(2.8)\quad |Rm|^2 = |W|^2 + \frac{4}{m-2}|Ric|^2 - \frac{2}{(m-1)(m-2)}S^2.\]

**Proof.** We first note that for any symmetric 2-tensor field $T$,

$$|T\otimes g|^2 = \langle T\otimes g, T\otimes g \rangle = 4\langle T, c(T\otimes g) \rangle$$  
$$= 4\langle T, (m-2)T + \text{Tr}(T)g \rangle$$  
$$= 4(m-2)|T|^2 + 4\text{Tr}(T)^2.$$  

In particular,

$$|E\otimes g|^2 = 4(m-2)|E|^2,$$

$$|g\otimes g|^2 = 4(m-2)m + 4m^2 = 8m(m-1).$$  

Since the decomposition (2.7) is orthogonal, we get

$$|Rm|^2 = |W|^2 + \frac{4}{m-2}|E|^2 + \frac{2}{m(m-1)}S^2.$$  

Finally we use

$$|E|^2 = \langle Ric - \frac{S}{m}g, Ric - \frac{S}{m}g \rangle$$  
$$= |Ric|^2 - \frac{2S}{m}|Ric, g| + \frac{S^2}{m^2}|g|^2$$  
$$= |Ric|^2 - \frac{2S}{m} + \frac{S^2}{m}$$  
$$= |Ric|^2 - \frac{S^2}{m}$$  

to get

$$|Rm|^2 = |W|^2 + \frac{4}{m-2}|Ric|^2 - \frac{2}{(m-1)(m-2)}S^2.$$  

□