LECTURE 7: DECOMPOSITION OF THE RIEMANN CURVATURE TENSOR

1. Some tensor algebra

Let V be any vector space. Recall that $\wedge^2 V \subset \otimes^2 V$ represents the space of anti-symmetric 2-tensors on V, while $S^2 V \subset \otimes^2 V$ represents the space of symmetric 2-tensors on V. Any 2-tensor can be decomposed uniquely as the summation of a symmetric 2-tensor and an anti-symmetric 2-tensor. If $\dim V = m$, then

$$\dim \wedge^2 V = \frac{m(m-1)}{2}$$

and

$$\dim S^2 V = \frac{m(m+1)}{2}.$$

As a consequence, $S^2(\wedge^2 V^*)$ contains 4-tensors that is symmetric with respect to $(1,2) \leftrightarrow (3,4)$ and is anti-symmetric with respect to $1 \leftrightarrow 2$ and with respect to $3 \leftrightarrow 4$, i.e.

$$T(X,Y,Z,W) = -T(Y,X,Z,W) = -T(X,Y,W,Z) = T(Z,W,X,Y).$$

It is a vector space of dimension

$$\dim S^2(\wedge^2 V^*) = \frac{m(m-1)(m^2-m+2)}{8}. \tag{1.1}$$

We note that the space of 4-forms, $\wedge^4 V^*$, is a subspace of $S^2(\wedge^2 V^*)$ with dimension

$$\dim \wedge^4 V^* = \binom{m}{4}. \tag{1.2}$$

Another way to describe the space $S^2(\wedge^2 V^*)$ is the following: Let $\alpha, \beta \in \wedge^2 V^*$ be any two 2-forms, both viewed as skew-symmetric 2-tensors. Recall that the *symmetric product* of α and β is a 4-tensor given by

$$(\alpha \odot \beta)(X,Y,Z,W) = \alpha(X,Y)\beta(Z,W) + \alpha(Z,W)\beta(X,Y). \tag{1.3}$$

Using a basis one has

$$(\alpha \odot \beta)_{ijkl} = \alpha_{ij}\beta_{kl} + \alpha_{kl}\beta_{ij}.$$

Obviously $\alpha \odot \beta$ is in $S^2(\wedge^2 V^*)$. Conversely, by dimension counting it is not hard to see that any element in $S^2(\wedge^2 V^*)$ can be written as a linear combination of elements of the form $\alpha \odot \beta$.

For any $T \in S^2(\wedge^2 V^*)$, we define its *Bianchi symmetrization* to be the 4-tensor

$$bT(X,Y,Z,W) = \frac{1}{3}(T(X,Y,Z,W) + T(Y,Z,X,W) + T(Z,X,Y,W)). \tag{1.4}$$

Lemma 1.1. *$b(\alpha \odot \beta) = \frac{1}{3}\alpha \wedge \beta$.*

Proof. We do the calculation using a basis:

$$\begin{aligned}(b(\alpha \odot \beta))_{ijkl} &= \frac{1}{3}\left(\alpha_{ij}\beta_{kl} + \alpha_{kl}\beta_{ij} + \alpha_{jk}\beta_{il} + \alpha_{il}\beta_{jk} + \alpha_{ki}\beta_{jl} + \alpha_{jl}\beta_{ki}\right) \\ &= \frac{1}{3}\left(\alpha_{ij}\beta_{kl} - \alpha_{ik}\beta_{jl} + \alpha_{il}\beta_{jk} + \alpha_{jk}\beta_{il} - \alpha_{jl}\beta_{ik} + \alpha_{kl}\beta_{ij}\right) \\ &= \frac{1}{3}(\alpha \wedge \beta)_{ijkl}.\end{aligned}$$

□

As a consequence, we see that the Bianchi symmetrization map

$$b : S^2(\wedge^2 V^*) \to S^2(\wedge^2 V^*)$$

has image

$$\mathrm{Im}(b) = \wedge^4 V^*.$$

Moreover, by definition it is easy to check that b is a projection, i.e.

$$b^2 = b.$$

So from the standard linear algebra one has a direct sum decomposition

$$S^2(\wedge^2 V^*) = \mathrm{Ker}(b) \oplus \mathrm{Im}(b) = \mathrm{Ker}(b) \oplus \wedge^4 V^*.$$

Explicitly for any $T \in S^2(\wedge^2 V^*)$, the decomposition is

$$T = [T - b(T)] + b(T).$$

We shall denote

$$\mathscr{C} = \mathrm{Ker}(b). \tag{1.5}$$

It is a vector space of dimension

$$\dim \mathscr{C} = \frac{m(m-1)(m^2-m+2)}{8} - \binom{m}{4} = \frac{1}{12}m^2(m^2-1) \tag{1.6}$$

The elements in $\mathscr{C}$ are called *curvature-like* tensors.

Example. Let $T_1, T_2 \in S^2 V^*$. Consider their *Kulkarni-Nomizu product*

$$\begin{aligned}T_1 \owedge T_2(X,Y,Z,W) =& T_1(X,Z)T_2(Y,W) - T_1(Y,Z)T_2(X,W) \\ &- T_1(X,W)T_2(Y,Z) + T_1(Y,W)T_2(X,Z).\end{aligned} \tag{1.7}$$

It is a 4-tensor in $S^2(\wedge^2 V^*)$. It is easy to see $b(T_1 \owedge T_2) = 0$, i.e.

$$T_1 \owedge T_2 \in \mathscr{C}. \tag{1.8}$$

So they are all curvature-like tensors.

Now suppose the vector space V is endowed with an inner product $g(\cdot,\cdot) = \langle\cdot,\cdot\rangle$, so that one can identify V^* with V using the musical isomorphisms $\flat$ and $\sharp$. In particular, for any element T in $\mathscr{C}$ and for any vectors X, Y, Z, the linear map

$$T(Z, X, \cdot, Y) : V \to \mathbb{R},$$

ccan be viewed as an element in V^* and thus be identified with an vector $\sharp T(Z, X, \cdot Y)$ in V. Fixing X, Y but letting Z varies, one gets a linear map

$$Z \mapsto \sharp T(Z, X, \cdot, Y).$$

Definition 1.2. For any curvature like tensor $T \in \mathscr{C}$, the *Ricci contraction* $c(T)$ of T is the following 2-tensor:

$$c(T)(X, Y) := \mathrm{Tr}(Z \mapsto \sharp T(Z, X, \cdot, Y)). \tag{1.9}$$

Lemma 1.3. *$c(T)(X, Y) = c(T)(Y, X)$.*

Proof. Fix X and Y. Let $K : V \to V$ be the map

$$K(Z) = \sharp T(Z, X, \cdot, Y)$$

and let $\tilde{K} : V \to V$ be the map

$$\tilde{K}(Z) = \sharp T(\cdot, X, Z, Y)$$

Then for any $Z, W \in V$, one has

$$\langle K(Z), W\rangle = \langle \sharp T(Z, X, \cdot, Y), W\rangle = T(Z, X, W, Y)$$
$$\langle Z, \tilde{K}(W)\rangle = \langle Z, \sharp T(\cdot X, W, Y)\rangle = T(Z, X, W, Y).$$

So $\tilde{K}$ is the transpose of K. In particular, they have the same trace. But by definition, the trace of K is $c(T)(X, Y)$, while the trace of $\tilde{K}$ is $c(T)(Y, X)$ (since T is in $\mathscr{C} \subset S^2(\wedge^2 V^*)$). □

Another way to see the symmetry of $c(T)$ is to write down everything in a basis. If $v^1, \cdots, v^n$ be a basis of V^*, and we denote $g^{pq} = g(v^p, v^q)$, then one has [exercise!]

$$c(T)_{ij} = g^{pq} T_{ipjq} \tag{1.10}$$

Back to the Kulkarni-Nomizu product. Since g is a symmetric 2-tensor, one can define a map

$$\Psi : S^2 V^* \to \mathscr{C}, \quad T \mapsto T \owedge g. \tag{1.11}$$

As we explained in lecture 1, a metric g on V induces a metric on space of tensors. For example, if $e^1, \cdots, e^m$ is an orthonormal basis of V^* under the metric g, then one defines the induced metric on $\otimes^4 V^*$ by requiring that $\{e^{i_1} \otimes e^{i_2} \otimes e^{i_3} \otimes e^{i_4}\}$ be an orthonormal basis. Under the induced metric, one can show that Ψ is "almost" the adjoint of the Ricci contraction map c:

Lemma 1.4. *For any $S \in \mathscr{C}$ and any $T \in S^2 V^*$, one has*

$$\langle S, \Psi(T)\rangle = 4\langle c(S), T\rangle. \tag{1.12}$$

Proof. We calculate using the orthonormal basis $e^1, \cdots, e^m$:

$$\begin{aligned}\langle S, T \owedge g\rangle &= S_{ijkl}(T_{ik}g_{jl} - T_{jk}g_{il} - T_{il}g_{jk} + T_{jl}g_{ik}) \\ &= S_{ijkj}T_{ik} - S_{ijki}T_{jk} - S_{ijjl}T_{il} + S_{ijil}T_{jl} \\ &= 4S_{ijkj}T_{ik} \\ &= 4(c(S))_{ik}T_{ik} = 4\langle c(S), T\rangle.\end{aligned}$$

□

One can repeat the procedure above to define the trace of a symmetric2-tensor using the metric: If T is any symmetric 2-tensor, then one can regard it as a map

$$T(X, \cdot) : V \to \mathbb{R}$$

which can be identified with an element in V^*, and thus a vector $\sharp T(X, \cdot)$ in V. Again one gets a linear map

$$X \mapsto \sharp T(X, \cdot).$$

The trace of this map is a constant now.

Definition 1.5. The *trace* of $T \in S^2V^*$, $\mathrm{Tr}(T)$, is the scalar

$$\mathrm{Tr}(T) := \mathrm{trace}(X \mapsto \sharp T(X, \cdot)). \tag{1.13}$$

In a basis one has

$$\mathrm{Tr}(T) = g^{pq}T_{pq}. \tag{1.14}$$

Note that by using the inner product, one can also write this as

$$\mathrm{Tr}(T) = \langle T, g\rangle, \tag{1.15}$$

since the right hand side equals

$$\langle T, g\rangle = T_{ij}g_{kl}g^{ik}g^{jl} = T_{ij}\delta^i_l g^{jl} = T_{ij}g^{ij}.$$

Lemma 1.6. *For any symmetric 2-tensor $T \in S^2V^*$,*

$$c(\Psi(T)) = (m-2)T + \mathrm{Tr}(T)g. \tag{1.16}$$

Proof. Again we prove this using the orthonormal basis above. We have

$$\begin{aligned}c(T \owedge g)_{ij} &= g^{pq}(T_{ij}g_{pq} - T_{pj}g_{iq} - T_{iq}g_{pj} + T_{pq}g_{ij}) \\ &= mT_{ij} - T_{ij} - T_{ij} + \mathrm{Tr}(T)g_{ij}.\end{aligned}$$

□

As a result, we can prove

Proposition 1.7. *The map Ψ is injective for $m > 2$.*

Proof. Suppose $\Psi(T) = 0$. Then

$$\begin{aligned} 0 = \langle \Psi(T), \Psi(T) \rangle &= 4\langle c(\Psi(T)), T \rangle \\ &= 4\langle (m-2)T + \mathrm{Tr}(T)g, T \rangle \\ &= 4(m-2)|T|^2 + 4(\mathrm{Tr}(T))^2. \end{aligned}$$

It follows that $T = 0$ if $m > 2$. □

In particular,

Corollary 1.8. *Ψ is bijective if $m = 3$.*

Proof. Since in this case, both S^2V^* and $\mathscr{C}$ are of 6 dimensional. □

2. Decomposition of the Riemann curvature tensor

Now suppose (M, g) be a Riemannian manifold. As usual, we shall use $S^2\left(\wedge^2 T^*M\right)$ to represent the sub-bundle of $\otimes^4 T^*M$ whose fiber at p is $S^2\left(\wedge^2 T_p^*M\right)$. By the formulae above,

$$\dim S^2\left(\wedge^2 T_p^* M\right) = \frac{m(m-1)(m^2-m+2)}{8},$$

where m is the dimension of M. By this definition, a section $T \in \Gamma(S^2(\wedge^2 T^*M))$ is a $(0,4)$-tensor that satisfies

$$T(X,Y,Z,W) = -T(Y,X,Z,W) = -T(X,Y,W,Z) = T(Z,W,X,Y).$$

Example. Last time we showed Rm is in $\Gamma(S^2(\wedge^2 T^*M))$.

All the constructions in the previous section works for tensor fields without any difficulty. For example, the *Bianchi symmetrization* of $T \in \Gamma\left(S^2(\wedge^2 T^*M)\right)$ is

$$bT(X,Y,Z,W) = \frac{1}{3}(T(X,Y,Z,W) + T(Y,Z,X,W) + T(Z,X,Y,W)).$$

So b is now a map $b : \Gamma\left(S^2(\wedge^2 T^*M)\right) \to \Gamma\left(S^2(\wedge^2 T^*M)\right)$. According to the first Bianchi identity, $b(Rm) = 0$.

Definition 2.1. A tensor field $T \in \Gamma\left(S^2(\wedge^2 T^*M)\right)$ is called *curvature-like* if $T \in \mathrm{Ker}(b)$.

Definition 2.2. We will call the symmetric 2-tensor field

$$\mathrm{Ric} := c(Rm) \tag{2.1}$$

the *Ricci curvature tensor* of (M, g).

Definition 2.3. We will call the function

$$S := \mathrm{Tr}(Ric) \tag{2.2}$$

the *scalar curvature* of (M, g).

Definition 2.4. We will call the symmetric 2-tensor field

$$A := \frac{1}{m-2}\left(\mathrm{Ric} - \frac{S}{2(m-1)}g\right) \tag{2.3}$$

the *Schouten tensor* of (M, g).

Definition 2.5. We call the curvature-like tensor

$$W = Rm - A \owedge g \tag{2.4}$$

the *Weyl curvature tensor* of (M, g).

Lemma 2.6. $c(W) = 0$.

Proof. Since $\mathrm{Tr}(g) = \langle g, g\rangle = m$, we have

$$\mathrm{Tr}(A) = \frac{1}{m-2}\left(S - \frac{S}{2(m-1)}m\right) = \frac{S}{2(m-1)}.$$

So

$$\begin{aligned} c(W) = \mathrm{Ric} - c(A \owedge g) &= \mathrm{Ric} - (m-2)A - \mathrm{Tr}(A)g \\ &= \mathrm{Ric} - (\mathrm{Ric} - \frac{S}{2(m-1)}g) - \frac{S}{2(m-1)}g \\ &= 0. \end{aligned}$$

□

Corollary 2.7. *If $m = 3$, then $W = 0$.*

Proof. Since $m = 3$, there exists symmetric 2-tensor field T so that $W = \Psi(T)$. So by lemma 1.4,

$$\langle W, W\rangle = \langle W, \Psi(T)\rangle = 4\langle c(W), T\rangle = 0.$$

So $W = 0$. □

Let's play with the decomposition

$$Rm = W + A \owedge g \tag{2.5}$$

a bit more.

Definition 2.8. We will call

$$E = Ric - \frac{S}{m}g \tag{2.6}$$

the *traceless Ricci tensor.*

It is called traceless because

$$\mathrm{Tr}E = \mathrm{Tr}(Ric) - \frac{S}{m}\mathrm{Tr}(g) = S - \frac{S}{m}m = 0.$$

Using E we can rewrite the equation (2.5) as

$$Rm = W + \frac{1}{m-2} E \owedge g + \frac{S}{2m(m-1)} g \owedge g \tag{2.7}$$

Proposition 2.9. *The decomposition (2.7) is an orthogonal decomposition.*

Proof. We have

$$\langle W, E \owedge g \rangle = \langle W, \Psi(E) \rangle = 4 \langle c(W), E \rangle = 0.$$

Similarly one can prove $\langle W, g \owedge g \rangle = 0$. Finally

$$\langle E \owedge g, g \owedge g \rangle = \langle E, c(g \owedge g) \rangle = \langle E, (2m-2) g \rangle = (2m-2) \mathrm{Tr}(E) = 0.$$

□

Our final goal for today is to prove

Theorem 2.10. *We have*

$$|Rm|^2 = |W|^2 + \frac{4}{m-2} |Ric|^2 - \frac{2}{(m-1)(m-2)} S^2. \tag{2.8}$$

Proof. We first note that for any symmetric 2-tensor field T,

$$\begin{aligned} |T \owedge g|^2 = \langle T \owedge g, T \owedge g \rangle &= 4 \langle T, c(T \owedge g) \rangle \\ &= 4 \langle T, (m-2) T + \mathrm{Tr}(T) g \rangle \\ &= 4(m-2)|T|^2 + 4 \mathrm{Tr}(T)^2. \end{aligned}$$

In particular,

$$\begin{aligned} |E \owedge g|^2 &= 4(m-2)|E|^2, \\ |g \owedge g|^2 &= 4(m-2)m + 4m^2 = 8m(m-1). \end{aligned}$$

Since the decomposition (2.7) is orthogonal, we get

$$|Rm|^2 = |W^2| + \frac{4}{m-2}|E|^2 + \frac{2}{m(m-1)} S^2.$$

Finally we use

$$\begin{aligned} |E|^2 &= \langle Ric - \frac{S}{m} g, Ric - \frac{S}{m} g \rangle \\ &= |Ric|^2 - \frac{2S}{m} \langle Ric, g \rangle + \frac{S^2}{m^2} |g|^2 \\ &= |Ric|^2 - \frac{2S^2}{m} + \frac{S^2}{m} \\ &= |Ric|^2 - \frac{S^2}{m} \end{aligned}$$

to get

$$|Rm|^2 = |W|^2 + \frac{4}{m-2} |Ric|^2 - \frac{2}{(m-1)(m-2)} S^2.$$

□