LECTURE 8: THE SECTIONAL AND RICCI CURVATURES

1. The Sectional Curvature

We start with some simple linear algebra. As usual we denote by $\otimes^2(\wedge^2 V^*)$ the set of 4-tensors that is anti-symmetric with respect to the first two entries and with respect to the last two entries.

Lemma 1.1. *Suppose $T \in \otimes^2(\wedge^2 V^*)$, $X, Y \in V$. Let $X' = aX + bY, Y' = cX + dY$, then*

$$T(X', Y', X', Y') = (ad - bc)^2 T(X, Y, X, Y).$$

Proof. This follows from a very simple computation:

$$\begin{aligned} T(X', Y', X', Y') &= T(aX + bY, cX + dY, aX + bY, cX + dY) \\ &= (ad - bc) T(X, Y, aX + bY, cX + dY) \\ &= (ad - bc)^2 T(X, Y, X, Y). \end{aligned}$$

□

Now suppose (M, g) is a Riemannian manifold. Recall that $\frac{1}{2} g \owedge g$ is a curvature-like tensor, such that

$$\frac{1}{2} g \owedge g(X_p, Y_p, X_p, Y_p) = \langle X_p, X_p \rangle \langle Y_p, Y_p \rangle - \langle X_p, Y_p \rangle^2.$$

Applying the previous lemma to Rm and $\frac{1}{2} g \owedge g$, we immediately get

Proposition 1.2. *The quantity*

$$K(X_p, Y_p) := \frac{Rm(X_p, Y_p, X_p, Y_p)}{\langle X_p, X_p \rangle \langle Y_p, Y_p \rangle - \langle X_p, Y_p \rangle^2}$$

depends only on the two dimensional plane $\Pi_p = \mathrm{span}(X_p, Y_p) \subset T_p M$, i.e. it is independent of the choices of basis $\{X_p, Y_p\}$ of Π_p.

Definition 1.3. We will call

$$K(\Pi_p) = K(X_p, Y_p)$$

the *sectional curvature* of (M, g) at p with respect to the plane Π_p.

Remark. The sectional curvature K is NOT a function on M (for $\dim M > 2$), but a function on the Grassmann bundle $G_{m,2}(M)$ of M.

Example. Suppose $\dim M = 2$, then there is only one sectional curvature at each point, which is exactly the well-known *Gaussian curvature* (exercise):

$$\kappa = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}.$$

In fact, for Riemannian manifold M of higher dimensions, $K(\Pi_p)$ is the Gaussian curvature of a 2-dimensional submanifold of M that is tangent to Π_p at p.

Example. For $(\mathbb{R}^m, g_0)$, one has $Rm = 0$, so

$$K(\Pi_p) \equiv 0.$$

Example. For (S^m, g_{round}), one has $Rm = \frac{1}{2} g \owedge g$, so

$$K(\Pi_p) \equiv 1.$$

So the sectional curvature is essentially the restriction of the Riemann curvature tensor to special set of vectors. A natural questions is: what information of Rm do we loss in this precedure? and the answer is: we don't loss any information, because of the symmetry of Rm. To show this, we first prove

Lemma 1.4. *Let T be a curvature-like tensor, and let*

$$f_{X,Y,Z,W}(t) = T(X+tZ, Y+tW, X+tZ, Y+tW) - t^2(T(X,W,X,W) + T(Z,Y,Z,Y)).$$

Then $(f_{X,Y,Z,W} - f_{Y,X,Z,W})''(0) = 12T(X,Y,Z,W)$.

Proof. Obviously $f_{X,Y,Z,W}$ is a polynomial of degree 4 in t, whose quadratic coefficients equals

$$T(Z,W,X,Y) + T(Z,Y,X,W) + T(X,W,Z,Y) + T(X,Y,Z,W).$$

which equals

$$2T(X,Y,Z,W) + 2T(Z,Y,X,W).$$

Similarly the quadratic coefficient of $f_{Y,X,Z,W}$ equals

$$2T(Y,X,Z,W) + 2T(Z,X,Y,W).$$

So the quadratic coefficient of $f_{X,Y,Z,W}(t) - f_{Y,X,Z,W}(t)$ is

$$2T(X,Y,Z,W) + 2T(Z,Y,X,W) - 2T(Y,X,Z,W) - 2T(Z,X,Y,W),$$

which, after applying the first Bianchi identity, equals

$$6T(X,Y,Z,W).$$

□

Applying the lemma to Rm, we get

Corollary 1.5. *The values of $Rm(X_p, Y_p, X_p, Y_p)$ for all $X_p, Y_p \in T_pM$ determines the tensor Rm at p.*

Of course for most Riemannian manifolds, its sectional curvatures are not constant and really depend on the 2-plane Π_p. We will explain the geometric meaning of $K(\Pi_p)$ later, after we develop more geometric tools. We will see whether the sectional curvatures of a Riemannian manifold are constant, or more generally although not constant but still bounded by some inequalities (e.g. non-negative, non-positive etc) will have much implications to the analysis, geometry and topology of (M, g).

Definition 1.6. A Riemannian manifold (M, g) is said to have *constant (sectional) curvature* if its sectional curvature $K(\Pi_p)$ is a constant, i.e. is independent of p and is independent of $\Pi_p \subset T_pM$.

Definition 1.7. We say (M, g) is a *flat manifold* if its sectional curvatures are identically zero.

Example. $(\mathbb{R}^m, g_0)$ is flat.

Example. The torus $\mathbb{T}^m = S^1 \times \cdots \times S^1$, endowed with the product metric of the standard rotation-invariant metric on S^1, is flat. To see this, one notice that near each point of $\mathbb{T}^m$, there is a local coordinates $\theta_1, \cdots, \theta_m$ so that

$$g = d\theta_1 \otimes d\theta_1 + \cdots + d\theta_m \otimes d\theta_m.$$

Since the coefficients are constants, one immediately see that Γ_{ij}^k's are zero, and thus R_{ijkl},s are zero. [One can also think of $T^m = \mathbb{R}^m/\mathbb{Z}^m$, then the flat metric on $\mathbb{T}^m$ we just described is the quotient metric of the standard flat metric on $\mathbb{R}^m$.]

Example. We have seen (S^m, g_{round}) has constant curvature 1. We will see in exercise that the hyperbolic space $(\mathbb{H}^m, g_{hyperbolic})$ has constant curvature -1.

Remark. In fact $(\mathbb{R}^m, g_0)$, (S^m, g_{round}) and $(\mathbb{H}^m, g_{hyperbolic})$ are *the canonical examples* of constant curvature manifolds, in the sense that the universal covering of any complete Riemannian manifold of constant curvature must be isometric to one of the three examples.

Proposition 1.8. *A Riemannian manifold has constant curvature k if and only if*

$$Rm = \frac{k}{2} g \owedge g.$$

Proof. According to lemma 1.4, if T is a curvature like tensor, then

$$T \equiv 0 \Longleftrightarrow T(X, Y, X, Y) = 0, \forall X, Y.$$

Apply this to the curvature-like tensor $T = Rm - \frac{k}{2} g \owedge g$, we see

$$Rm = \frac{k}{2} g \owedge g \Longleftrightarrow K(\Pi_p) = k, \forall\ \Pi_p.$$

$\square$

On the other hand side, last time we proved that the Riemann curvature tensor admits the following *orthogonal* (thus unique!) decomposition

$$Rm = W + \frac{1}{m-2} E \owedge g + \frac{S}{2m(m-1)} g \owedge g$$

From this we get (Recall $E = Ric - \frac{S}{m} g$)

Proposition 1.9. *A Riemannian manifold (M, g) has constant curvature k if and only if $W = 0$ and $Ric = (m-1)kg$.*

Proof. If (M, g) has constant sectional curvature k, then by the uniqueness of the decomposition,

$$W = 0, \quad E = 0 \quad \text{and} \quad S = m(m-1)k.$$

As a result,

$$Ric = (m-1)kg.$$

Conversely if $W = 0$ and $Ric = (m-1)kg$, then

$$S = Tr(Ric) = m(m-1)k.$$

So $E = 0$ and thus

$$Rm = \frac{k}{2} g \owedge g,$$

i.e. (M, g) has constant curvature k. □

We remark that as a consequence, the scalar curvature of for a Riemannian manifold of constant curvature k must be

$$S = m(m-1)k.$$

The next theorem shows that for Riemannian manifolds of dimension ≥ 3, if the sectional curvature depends only on p, then it is independent of p. Before we prove it, we need the following

Lemma 1.10. *For any vector field V on M, we have $\nabla_V g \owedge g = 0$.*

Proof. The covariant derivative with respect to V of the first term in the expression of $g \owedge g$ is

$$\begin{aligned} V(g(X,Z)g(Y,W)) - g(\nabla_V X, Z)g(Y,W) - g(X, \nabla_V Z)g(Y,W) \\ - g(X,Z)g(\nabla_V Y, W) - g(X,Z)g(Y, \nabla_V W) \end{aligned}$$

which vanishes according to the metric compatibility of ∇. Same holds for the other three terms of $g \owedge g$. □

Now we are ready to prove

Theorem 1.11 (Schur)**.** *Let (M, g) be a Riemannian manifold of dimension $m \geq 3$. If $K(\Pi_p) = f(p)$ depends only on p, then (M, g) is of constant curvature.*

Proof. For simplicity we denote $R_0 = \frac{1}{2} g \owedge g$. Then by the assumption, $Rm = fR_0$, where f is a function on M. Since $\nabla_V R_0 = 0$ for all V, we conclude

$$\nabla_V Rm = \nabla_V(fR_0) = V(f)R_0.$$

Now let's apply the second Bianchi identity:

$$\begin{aligned} 0 &= (\nabla_X Rm)(Y,Z,W,V) + (\nabla_Y Rm)(Z,X,W,V) + (\nabla_Z Rm)(X,Y,W,V) \\ &= X(f)R_0(Y,Z,W,V) + Y(f)R_0(Z,X,W,V) + Z(f)R_0(X,Y,W,V). \end{aligned}$$

This identity holds for any X,Y,Z,W,V, and the right hand side only depends on X_p, Y_p, Z_p, W_p, V_p since it is a tensor identity. If we take $X_p, Y_p, Z_p \in T_pM$ so that (note that this is only possible if $\dim M \geq 3$)

$$X_p \neq 0, \quad Y_p \neq 0, \quad |Z_p| = 1, \quad \langle X_p, Y_p \rangle = \langle Y_p, Z_p \rangle = \langle Z_p, X_p \rangle = 0$$

and let $V_p = Z_p$, then the above identity becomes

$$X_p(f)\langle Y_p, W_p \rangle - Y_p(f)\langle X_p, W_p \rangle = 0.$$

This holds for any $W_p \in T_pM$, so we conclude

$$X_p(f)Y_p - Y_p(f)X_p = 0$$

holds as long as $X_p, Y_p \neq 0$ and $\langle X_p, Y_p \rangle = 0$. In particular, we know that

$$X_p(f) = 0$$

for all $X_p \neq 0$. This is true for all p. So f must be a constant function on M. $\square$

Remark. Obviously the theorem fails in dimension 2, in which case the sectional curvature is always a function on M but need not be a constant.

2. The Ricci Curvatures

Recall that the Ricci curvature tensor Ric is the contraction of the Riemann curvature tensor Rm,

$$Ric(X,Y) = c(Rm)(X,Y) = \mathrm{Tr}(Z \mapsto \sharp Rm(X,Z,Y,\cdot)).$$

It is a symmetric $(0,2)$-tensor field on M. In local coordinates one has

$$Ric_{ij} = g^{pq} R_{ipjq}$$

One can also rewrite the previous formulae using the $(1,3)$-tensor R,

$$Ric(X,Y) = \mathrm{Tr}(Z \mapsto R(X,Z)Y)$$

and locally

$$Ric_{ij} = R_{ipj}{}^{p}.$$

Definition 2.1. For any unit tangent vector $X_p \in S_pM \subset T_pM$, we call

$$Ric(X_p) = Ric(X_p, X_p)$$

the *Ricci curvature* of M at p in the direction of X_p.

So again the Ricci curvature function Ric is not a function on M, but a function on the unit sphere bundle $SM \subset TM$. Alternatively, one can think of the Ricci curvature as a function defined on one-dimensional subspaces of T_pM. Since in the definition of the Ricci curvature we only use the information of the Ricci tensor Ric on special set of vectors, one would like to know what information do we lost, and again by symmetry of Ric we don't lost anything. This follows from the following obvious lemma:

Lemma 2.2. *Let T be a symmetric 2-tensor. Then for any X, Y,*

$$T(X,Y) = \frac{1}{2}\left(T(X+Y,X+Y) - T(X,X) - T(Y,Y)\right).$$

Applying the lemma to Ric, we get the following expression of the Ricci tensor in terms of the Ricci curvature:

Corollary 2.3. *For any $Y_p \neq -X_p$ we have*

$$\mathrm{Ric}(X_p,Y_p) = \frac{1}{2}\left[\|X_p+Y_p\|^2 Ric(\widehat{X_p+Y_p}) - \|X_p\|^2 Ric(\widehat{X}_p) - \|Y_p\|^2 Ric(\widehat{X}_p)\right],$$

where we denoted $\widehat{X} = X/\|X\|$.

Similarly we can define the conception like a space of constant Ricci curvature, a space of non-negative Ricci curvature etc. In particular, we have

Proposition 2.4. *A Riemannian manifold has constant Ricci curvature k if and only if*

$$Ric = kg.$$

Proof. Apply lemma 2.2 to the symmetric tensor $Ric - kg$. □

We also have the following version of Schur's theorem (which actually implies the Schur's theorem for sectional curvature that we just proved):

Theorem 2.5 (Schur)**.** *Let (M, g) be a Riemannian manifold of dimension $m \geq 3$. If $Ric(X_p) = f(p)$ depends only on p, then (M, g) has constant Ricci curvature.*

Proof. Apply the second Bianchi identity. Exercise. □