LECTURE 9: MORE ON CURVATURES

1. Various types of special Riemannian manifolds

We shall briefly describe various classes of Riemannian manifolds which are special in terms of curvatures.

1.1. **Flat manifolds.**

Definition 1.1. We say a Riemannian manifold (M, g) is *locally Euclidean* if near each point $p \in M$, there is a coordinate system $(U, x^1, \cdots, x^m)$ so that on U,

$$g = dx^1 \otimes dx^1 + \cdots dx^m \otimes dx^m.$$

For example, the standard torus $\mathbb{T}^m$, either viewed as the product $S^1 \times \cdots \times S^1$ (endow each S^1 with the standard rotation-invariant metric $d\theta \otimes d\theta$), or viewed as the quotient $\mathbb{R}^m/\mathbb{Z}^m$ of the standard Euclidean space $\mathbb{R}^m$, is locally Euclidean with an obvious choice of local coordinates.

Obviously if (M, g) is locally Euclidean, then the Christoffel symbols associated to the coordinate system alluded above are all zero, thus the curvature tensor of (M, g) vanishes. As a result, (M, g) must be a flat manifold. Conversely, we have

Theorem 1.2. *If (M, g) is flat if and only if it is a locally Euclidean space.*

Sketch of proof. One need to prove that any flat manifold (M, g) is locally Euclidean.

By using the geometric method that we will develop next time, one can prove that if (M, g) is flat, then one can find, in some neighborhood U of each point p a *flat* frame, i.e. a set of vector fields $X_1, \cdots, X_m$ on U such that

- $\{X_i(q) \mid 1 \le i \le m\}$ form a basis of T_qM for every $q \in U$,
- $\nabla_Y X_i = 0$ for all i and for all vector field Y.

This fact has two consequences:

First by torsion freeness, one gets

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0$$

for all $1 \le i, j \le m$. So by the Frobenius theorem in manifold theory, after shrinking U if necessary, one can find local coordinates $\{x^1, \cdots, x^m\}$ near p so that

$$X_i = \partial_i, \quad 1 \le i \le m.$$

Secondly by metric compatibility, for any vector field Y on U,

$$Y g_{ij} = Y\langle X_i, X_j\rangle = \langle \nabla_Y X_i, X_j\rangle + \langle X_i, \nabla_Y X_j\rangle = 0.$$

So each g_{ij} is a constant on U.

So the metric matrix (g_{ij}) is a constant (positive definite) matrix on U. By a coordinate change one can make each $g_{ij} = \delta_{ij}$. So (M, g) is locally Euclidean. $\square$

Remark. More generally, if (M, g) has constant sectional curvature k, then (M, g) is locally isometric to

- $(S^m, \frac{1}{k} g_{round})$ if $k > 0$,
- $(\mathbb{R}^m, g_0)$ if $k = 0$,
- $(H^m, -\frac{1}{k} g_{hyperbolic})$ if $k < 0$,

1.2. **Einstein manifolds.**

Definition 1.3. We say (M, g) is an *Einstein manifold* if there exists constant λ such that

$$Ric = \lambda g.$$

Last time we have seen that (M, g) is an Einstein manifold if and only if it has constant Ricci curvature λ. Moreover, since $\mathrm{Tr}(Ric) = S$ (the scalar curvature) and $\mathrm{Tr}(g) = m$ (the dimension of M), we conclude that the constant λ for Einstein manifold must be

$$\lambda = \frac{S}{m}.$$

As a consequence, the traceless Ricci tensor for an Einstein manifold must vanish:

$$E = Ric - \frac{S}{m} g = \frac{S}{m} g - \frac{S}{m} g = 0.$$

Example. Suppose (M, g) has constant curvature k, then last time we showed

$$Ric = c(Rm) = (m-1)kg.$$

So any constant sectional curvature manifold must be an Einstein manifold. Conversely if (M, g) is an Einstein manifold and $W = 0$, then (M, g) has constant sectional curvature. [See proposition 1.9 in lecture 8.] Thus we get

Proposition 1.4. *For $m = 2$ or 3, (M, g) is Einstein if and only if (M, g) has constant sectional curvature.*

Proof: (Although it is implied by proposition 1.9 in lecture 8, we give a direct proof here.) For $m = 2$, we have

$$Rm = \frac{K(x)}{2} g \owedge g,$$

where $K(x)$ is the sectional curvature at x. As a result,

$$Ric = c(Rm) = (m-1)K(x)g.$$

So (M, g) is an Einstein manifold if and only if $K(x)$ is a constant.

For $m = 3$, we have $W = 0$. If (M, g) is Einstein we also have $E = 0$. So

$$Rm = \frac{S}{12} g \owedge g,$$

This implies that the sectional curvature of (M, g) is the constant $\frac{S}{6} = \frac{\lambda}{2}$. □

Remark. From the proof we get

- For $m = 2$, Rm is determined by S.
- For $m = 3$, Rm is determined by Ric.

So to find an Einstein manifold that is not of constant curvature, one must look at manifolds with dimension at least 4. To discover which manifold admits Einstein metric and which does not is still a very active research topic today. Here is an example:

Example. Let $M = S^2 \times S^2$, equipped with the standard product metric $g = g_{S^2} \oplus g_{S^2}$. A better way to write down this metric is

$$g = \pi_1^* g_{S^2} + \pi_2^* g_{S^2}.$$

Note that S^2 has constant curvature 1, so that it is Einstein and

$$Ric(g_{S^2}) = g_{S^2}.$$

It follows

$$Ric(g) = \pi_1^* Ric(g_{S^2}) + \pi_2^* Ric(g_{S^2}) = \pi_1^* g_{S^2} + \pi_2^* g_{S^2} = g.$$

This shows that M is an Einstein manifold.

Obviously M is not of constant sectional curvature. In fact, for $p = (p_1, p_2) \in S^2 \times S^2$, if we let e_1, e_2 be a basis of $T_{p_1} S^2$ and let e_3, e_4 be a basis of $T_{p_2} S^2$, then

$$K(d\iota_q^1(e_1), d\iota_q^1(e_2)) = 1, \qquad K(d\iota_q^1(e_1), d\iota_p^1(e_3)) = 0,$$

where $\iota_q : S^2 \to S^2 \times S^2$ is the embedding that maps p to (p, q), while $\iota_p : S^2 \to S^2 \times S^2$ is the embedding that maps q to (p, q).

One can also prove that the sectional curvature of $S^2 \times S^2$ is not constant by calculating the Weyl tensor of g and showing that it is nonzero. In fact, one has

$$W(g) = \frac{1}{3}(\pi_1^* g_{S^2} \owedge \pi_1^* g_{S^2} + \pi_2^* g_{S^2} \owedge \pi_2^* g_{S^2} - \pi_1^* g_{S^2} \owedge \pi_2^* g_{S^2}).$$

I will leave this computation as an exercise.

Remark. In the previous example, the metric on $S^2 \times S^2$ has nonnegative sectional curvature. A long time open problem in Riemannian geometry is

Conjecture (Hopf Conjecture). $S^2 \times S^2$ admits no positive sectional curvature metric.

1.3. **Locally conformally flat manifolds.**

Definition 1.5. We say a Riemannian manifold (M, g) is *locally conformally flat* if for any $p \in M$, there is a neighborhood U of p and a smooth function $f \in C^\infty(U)$ so that the metric $\bar{g} = e^{2f} g$ is flat in U.

I will leave it as an exercise for you to prove

Lemma 1.6. *For $m = 2$, the sectional curvature $\overline{K}(x)$ of $\bar{g} = e^{2f} g$ and the sectional curvature $K(x)$ of g is related by*

$$\overline{K}(x) = e^{-2f}(K(x) - \Delta f).$$

In analysis, one can prove that for any smooth function K, the elliptic equation

$$\Delta f(x) = K(x)$$

is always solvable in an open set. So we immediately get

Theorem 1.7 (Gauss 1822)**.** *Any Riemannian manifold of dimension two is locally conformally flat.*

The following lemma says that the Weyl tensor is invariant under conformal change. The proof will be left as an exercise. [If we replace the (0,4) tensor W by a corresponding $(1, 3)$-tensor, then we do get invariance.]

Lemma 1.8. *If $\bar{g} = e^{2f} g$, then $\overline{W} = e^{2f} W$.*

Since flat manifolds has vanishing Weyl tensor, a direct consequence of the lemma is that if (M, g) is locally conformally flat, then the Weyl tensor equals zero. Conversely, one has

Theorem 1.9 (Weyl-Schouten)**.** *Let (M, g) be a Riemannian manifold.*

(1) *For $m = 3$, (M, g) is locally conformally flat if and only if*

$$(\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z) = 0$$

for all X, Y, Z, where A is the Schouten tenser.

(2) *For $m \geq 4$, (M, g) is locally conformally flat if and only if $W = 0$.*

The proof will be left as part of one possible final project.

As a consequence, we see

- $S^2 \times S^2$ with the standard metric is not locally conformally flat since $W \neq 0$.
- Any constant curvature space is locally conformally flat:
  - If $m = 2$, then (M, g) is locally conformally flat by Gauss theorem.
  - If $m \geq 4$, then (M, g) is locally conformally flat since $W = 0$.
  - If $m = 3$, we have $W = 0$ and thus $Rm = \Psi(A)$, where $\Psi(T) = T \owedge g$. Since (M, g) is of constant curvature, we have $Rm = \Psi(\frac{1}{2} g)$. But we know that the map Ψ is injective. So $A = \frac{k}{2} g$, which implies $\nabla A = 0$.

- $S^{m_1} \times H^{m_2}$ is locally conformally flat: we have
$$\begin{aligned} Rm &= \frac{1}{2}\pi_1^* g_1 \owedge \pi_1^* g_1 - \frac{1}{2}\pi_2^* g_2 \owedge \pi_1^* g_2 \\ &= \frac{1}{2}(\pi_1^* g_1 + \pi_2^* g_2) \owedge (\pi_1^* g_1 - \pi_2^* g_2) = \frac{1}{2}(\pi_1^* g_1 - \pi_2^* g_2) \owedge g. \end{aligned}$$
So $W = 0$ and $A = \frac{1}{2}(\pi_1^* g_1 - \pi_2^* g_2)$. As a consequence, $\nabla A = 0$.

1.4. **Manifolds with Constant scalar curvature.** According to the well known uniformization theorem in complex analysis, every surface has a conformal metric of constant Gaussian curvature, in other words, for any 2 dimensional Riemannian manifold (M, g), there is a function $f \in C^\infty(M)$ so that $(M, e^f g)$ has constant Gaussian(=sectional=Ricci=scalar) curvature.

A natural question is whether one can generalize the theorem to higher dimension. Obviously one cannot hope to find constant sectional curvature metric in the conformal class of most (M, g). In fact there exists topological obstructions that prohibits M to admit constant sectional curvature at all. People also found many topological restrictions for a (4 dimensional) manifold to admit a constant Ricci curvature metric.

As for scalar curvature, in 1960 H. Yamabe posed the following question:

The Yamabe problem: Given a compact Riemannian manifold (M, g) of dimension $m \geq 3$, does there exists a metric $\bar{g}$ conformal to g (i.e. $\bar{g} = e^{2f} g$) so that the scalar curvature of $\bar{g}$ is a constant on M?

The problem was answered positively (step by step) by H. Yamabe (1960), N. Trudinger (1968), T. Aubin (1976) and finally by R. Schoen (1984). In other words, for any compact Riemannian manifold, there exists lots of constant scalar curvature metrics!

For noncompact manifolds, the issue is more subtler.

2. Calculation with moving frames

Before we turn to the geometric theory on Riemannian manifold, let me briefly describe the Cartan's formulation of the theory of connections and curvatures, which is quite useful when doing computations, and is used in extending the theory to principle bundles.

As we mentioned, we can regard any linear connection ∇ (acting on vector fields) as a map
$$\nabla : \Gamma(TM) \to \Gamma(TM) \otimes \Gamma(T^*M).$$
So if $\{e_1, \cdots, e_m\}$ is a local frame of TM, then one can find one forms $\{\theta_i^j\}$ so that

(1) $$\nabla e_i = e_j \otimes \theta_i^j.$$

These θ_i^j's are called *connection 1-forms*, and they are only defined locally.

We shall denote by $\{\omega^1, \cdots, \omega^m\}$ the dual frame of T^*M to the given frame $\{e_1, \cdots, e_m\}$. According to our extension of linear connection on 1-forms, we know for any $X \in \Gamma(TM)$,

$$(\nabla_X \omega^i)(e_j) = X(\omega^i(e_j)) - \omega^i(\nabla_X e_j) = -\omega^i(\theta_j^k(X)e_k) = -\theta_j^i(X).$$

It follows that the linear connection ∇ acting on one forms, viewed as a map

$$\nabla : \Gamma(T^*M) \to \Gamma(T^*M) \otimes \Gamma(T^*M),$$

satisfies

$$\nabla \omega^i = -\omega^j \otimes \theta_j^i. \tag{2}$$

Using the connection 1-forms, one can calculate curvature tensor as follows:

$$\begin{aligned} R(X,Y)e_i &= -\nabla_X \nabla_Y e_i + \nabla_Y \nabla_X e_i + \nabla_{[X,Y]} e_i \\ &= -\nabla_X(\theta_i^j(Y)e_j) + \nabla_Y(\theta_i^j(X)e_j) + \theta_i^j([X,Y])e_j \\ &= -X(\theta_i^j(Y))e_j - \theta_i^j(Y)\theta_j^k(X)e_k + Y(\theta_i^j(X))e_j \\ &\quad + \theta_i^j(X)\theta_j^k(Y)e_k + \theta_i^j([X,Y])e_j \\ &= -(d\theta_i^j)(X,Y)e_j + \theta_i^k \wedge \theta_k^j(X,Y)e_j. \end{aligned}$$

As a consequence, we get

$$-d\theta_i^j + \theta_i^k \wedge \theta_k^j = R_{kli}{}^j \omega^k \otimes \omega^l = \frac{1}{2} R_{kli}{}^j \omega^k \wedge \omega^l. \tag{3}$$

We shall denote

$$\Omega_i^j = -\frac{1}{2} R_{kli}{}^j \omega^k \wedge \omega^l,$$

and call it the *curvature 2-form*. (Note: it is actually globally defined!) Using the curvature 2-form, one can rewrite (**??**) as

$$d\theta_i^j = \theta_i^k \wedge \theta_k^j + \Omega_i^j. \tag{4}$$

Now suppose the connection is torsion free. Then

$$\begin{aligned} d\omega^i(X,Y) &= X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i([X,Y]) \\ &= X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i(\nabla_X Y - \nabla_Y X) \\ &= (\nabla_X \omega^i)(Y) - (\nabla_Y \omega^i)(X). \end{aligned}$$

It follows

$$d\omega^i = \omega^j \otimes \theta_j^i - \theta_j^i \otimes \omega^j = \omega^j \wedge \theta_j^i. \tag{5}$$

If we assume further that ∇ is metric compatible, and thus is a Levi-Civita connection, and suppose $\{e_1, \cdots, e_m\}$ is an *orthonormal* frame, then

$$0 = \langle \nabla e_i, e_j \rangle + \langle e_i, \nabla e_j \rangle = \langle e_k \otimes \theta_i^k, e_j \rangle + \langle e_i, e_k \otimes \theta_j^k \rangle = \theta_i^j + \theta_i^j. \tag{6}$$

In particular, we have $\Omega_i^j = -\Omega_j^i$.

Let's summarize:

> Let ∇ be a linear connection on M. Suppose $\omega^1, \cdots, \omega^m$ is a local frame of T^*M, then
>
> - The connection 1-forms θ_i^j's are the one-forms such that
> $$\nabla \omega^i = -\omega^j \otimes \theta_j^i.$$
> - The curvature 2-form Ω_i^j's are given by
> $$\Omega_i^j = d\theta_i^j - \theta_i^k \wedge \theta_k^j.$$
> - If ∇ is torsion-free, then
> $$d\omega^i = \omega^j \otimes \theta_j^i - \theta_j^i \otimes \omega^j = \omega^j \wedge \theta_j^i.$$
> - If ∇ is compatible with a Riemannian metric g, and if $\{\omega^1, \cdots, \omega^m\}$ is an orthonormal basis, then
> $$\theta_i^j + \theta_i^j = 0.$$

One can use Cartan's formulation to set up the whole theory that we have learned. To illustrate, let's give a couple examples.

Example 1: Calculating curvatures

Let $\{e_1, \cdots, e_m\}$ be an orthonormal frame, the sectional curvature of the plane spanned by $\{e_i, e_j\}$ is

$$K(e_i, e_j) = Rm(e_i, e_j, e_i, e_j) = R_{ijij} = R_{iji}{}^j = \Omega_i^j(e_j, e_i). \tag{7}$$

Theorem 2.1. *(M, g) has constant curvature c if and only if for any orthonormal frame $\{e_i\}$, we have*

$$\Omega_j^i = c\omega^i \wedge \omega^j. \tag{8}$$

Proof. Suppose (**??**) holds for any orthonormal frame. Let Π_p be any two dimensional plane in T_pM. Choose an orthonormal basis $\{e_1, e_2\}$ of Π_p, extend it to an orthonormal frame and denote by $\omega^1, \cdots, \omega^m$ the dual frame. Then by (**??**) we see

$$K(\Pi_p) = c.$$

Conversely suppose (M, g) has constant sectional curvature c, then with respect to any orthonormal frame,

$$R_{ijk}{}^l = R_{ijkl} = c(\delta_{jk}\delta_i^l - \delta_{ik}\delta_j^l).$$

This implies (**??**). □

Example. Consider the upper half space $\mathbb{H}^m$ with the hyperbolic metric

$$g_{hyperbolic} = \frac{1}{(x^m)^2}(dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m).$$

One choose an orthonormal basis $\{e_i = x^m \partial_i\}$ and thus the dual basis $\{\omega^i = \frac{1}{x^m} dx^i\}$. Then

$$\omega^j \wedge \theta_j^i = d\omega^i = -\frac{1}{(x^m)^2} dx^m \wedge dx^i = -\omega^m \wedge \omega^i. \tag{9}$$

We denote $\theta_j^i = c_{jk}^i \omega^k$. [Note: They are NOT Christoffel's symbols!] Note by the metric compatibility, we have $\theta_j^i = -\theta_i^j$, which implies

$$c_{jk}^i = -c_{ik}^j.$$

In particular, $c_{ik}^i = 0$. Plug the expression of θ_j^i into (**??**) we get, for $\{j,k\} \neq \{m,i\}$,

$$c_{jk}^i = c_{kj}^i$$

This implies, for $i \neq j$ and $i, j \neq m$,

$$c_{ik}^j = c_{ki}^j = -c_{ji}^k = -c_{ij}^k = c_{kj}^i = c_{jk}^i = -c_{ik}^j,$$

It follows $c_{ik}^j = 0$ for all $i, j < m$, i.e.

$$\theta_j^i = 0, \quad i, j < m.$$

Put this into (**??**) we see, for $i < m$,

$$\theta_m^i = -\omega^i.$$

So we get for $i, j < m$

$$\Omega_j^i = -\theta_j^k \wedge \theta_k^i = -\omega^i \wedge \omega^j$$

and for $i < m$

$$\Omega_m^i = d\theta_m^i = -d\omega^i = -\omega^i \wedge \omega^m.$$

This implies that the hyperbolic space has constant curvature -1.

Example 2: Proving the Bianchi identities.

To get the first Bianchi identity, we just differentiate:

$$\begin{aligned} 0 = d^2\omega^i &= d\omega^j \wedge \theta_j^i - \omega^j \wedge d\theta_j^i \\ &= \omega^k \wedge \theta_k^j \wedge \theta_j^i - \omega^j \wedge (\theta_j^k \wedge \theta_k^i + \Omega_j^i) \\ &= -\omega^j \wedge \Omega_j^i \\ &= \frac{1}{2} R_{klj}{}^i \omega^j \wedge \omega^k \wedge \omega^l \\ &= \sum_{j<k<l} \left(R_{klj}{}^i + R_{ljk}{}^i + R_{jkl}{}^i \right) \omega^j \wedge \omega^k \wedge \omega^l. \end{aligned}$$

As a consequence, we get for distinct k, l, j's,

$$R_{klj}{}^i + R_{ljk}{}^i + R_{jkl}{}^i = 0.$$

If two or three of k, l, j's are the same, then the first Bianchi identity trivial.

Similarly one can prove the second Bianchi identity by studying $d^2\theta_i^j = 0$.