

LECTURE 10: THE PARALLEL TRANSPORT

1. THE PARALLEL TRANSPORT

We shall start with the geometric meaning of linear connections. Suppose M is a smooth manifold with a linear connection ∇ . Let $\gamma : [a, b] \rightarrow M$ be an embedded smooth curve in M , and let X be any vector field on M . We will call

$$E(X, \gamma) = \int_a^b \|\nabla_{\dot{\gamma}(t)} X\|^2 dt.$$

the *energy* of X along γ .

Definition 1.1. We say a vector field X is *parallel* along γ if

$$\nabla_{\dot{\gamma}(t)} X = 0$$

for all t .

Example. Let $M = \mathbb{R}^n$ with the standard Euclidean space, with standard linear connection ∇ such that $\nabla_X Y = X(Y^j)\partial_j$. Let γ be any curve and X be a vector field. Recall that as a vector at $\gamma(t)$, $\dot{\gamma}(t) = d\gamma(\frac{d}{dt})$. So for X to be parallel along γ , we need

$$0 = \nabla_{\dot{\gamma}(t)} X = \dot{\gamma}(t)(X^i)\partial_i = \frac{d}{dt}(X^i \circ \gamma)\partial^i.$$

It follows that X is parallel along γ if and only if X^i 's are constants on γ , i.e. if and only if X is a constant vector field along γ .

Theorem 1.2. For any curve $\gamma : [a, b] \rightarrow M$, any $t_0 \in [a, b]$ and any $X_0 \in T_{\gamma(t_0)}M$, there exists a unique vector field X along γ which is parallel, such that $X(\gamma(t_0)) = X_0$.

Proof. [It is enough to prove the theorem for the case when the curve lies in one coordinate patch, since the general case follows from this local existence/uniqueness and a standard compactness argument.] Suppose in a local coordinate system, $X_0 = X_0^j \partial_j|_{\gamma(t_0)}$. To find the parallel vector field $X = X^j \partial_j$, we need to solve the equation

$$0 = \nabla_{\dot{\gamma}(t)}(X^j \partial_j) = \frac{dX^j(\gamma(t))}{dt} \partial_j + X^j(\gamma(t)) \nabla_{\dot{\gamma}(t)} \partial_j$$

If we let $f_j(t) = X^j(\gamma(t))$ and let $a_{ij}(t)$ be such that $\nabla_{\dot{\gamma}(t)} \partial_k = \sum a_{kj}(t) \partial_j$, then we get a system of linear ODEs

$$f'_j(t) + \sum_k f_k(t) a_{kj}(t) = 0, \quad 1 \leq j \leq m$$

with initial conditions $f_j(t_0) = X_0^j$. Now apply the classical existence and uniqueness results in ODE. \square

Definition 1.3. We will call the map

$$P_{t_0,t}^\gamma : T_{\gamma(t_0)}M \rightarrow T_{\gamma(t)}M, \quad X_0 = X(\gamma(0)) \mapsto X(\gamma(t))$$

the *parallel transport* from $\gamma(t_0)$ to $\gamma(t)$ along γ , where X is the parallel vector field along γ such that $X(\gamma(t_0)) = X_0$.

Remark. Any immersed curve can be divided into pieces such that each piece is an embedded curve. So the parallel transport can be defined along immersed curves.

Lemma 1.4. Any parallel transport $P_{t_0,t}^\gamma$ is a linear isomorphism.

Proof. The linearity comes from the fact that the solution of a homogeneous linear ODE system depends linearly on initial data (the superposition principle).

$P_{t_0,t}^\gamma$ is invertible since

$$P_{t_0,t}^\gamma P_{a+b-t,t_0}^{-\gamma} = \text{Id},$$

where $-\gamma$ is the “opposite curve” $(-\gamma)(s) = \gamma(a+b-s)$. \square

We just see that from any linear connection one gets families of parallel transports which are isomorphisms. Conversely, ∇ is totally determined by its parallel transports.

Proposition 1.5. Let $\gamma : [a, b] \rightarrow M$ be a smooth curve on M such that $\gamma(t_0) = p$ and $\dot{\gamma}(t_0) = X_0 \in T_pM$. Then for any vector field $Y \in \Gamma(TM)$,

$$\nabla_{X_0} Y(p) = \lim_{t \rightarrow t_0} \frac{(P_{t_0,t}^\gamma)^{-1}(Y(\gamma(t))) - Y(\gamma(t_0))}{t - t_0}$$

Proof. Let $\{e_1, \dots, e_m\}$ be a basis of T_pM . Let $e_i(t) = P_{t_0,t}^\gamma(e_i)$. Then by the previous lemma, $\{e_1(t), \dots, e_m(t)\}$ is a basis of $T_{\gamma(t)}M$. So there exist functions $X^i(t)$ along curve γ so that $Y(\gamma(t)) = Y^i(t)e_i(t)$. It follows that

$$(P_{t_0,t}^\gamma)^{-1}(Y(\gamma(t))) = Y^i(t)e_i.$$

So

$$\lim_{t \rightarrow t_0} \frac{(P_{t_0,t}^\gamma)^{-1}(Y(\gamma(t))) - Y(p)}{t - t_0} = \lim_{t \rightarrow t_0} \frac{Y^i(t)e_i - Y(p)}{t - t_0} = \dot{Y}^i(t_0)e_i.$$

On the other hand side,

$$\nabla_{X_0} Y(p) = (\nabla_{X_0} Y^i)(p)e_i + Y^i(t_0)\nabla_{X_0} e_i(t_0) = \dot{Y}^i(t_0)e_i,$$

so the conclusion follows. \square

Remark. We used linear connection to define parallel transport. One could also use parallel transport to define its linear connection, i.e. one can define a parallel transport to be a family of linear isomorphisms $\{P^\gamma\}$ that satisfies several axioms:

- (1) $P^{\gamma_1\gamma_2} = P^{\gamma_1} \circ P^{\gamma_2}$.
- (2) $P^{-\gamma} = (P^\gamma)^{-1}$.
- (3) P^γ depends smoothly on γ .
- (4) If $\gamma_1(0) = \gamma_2(0)$, $\dot{\gamma}_1(0) = \dot{\gamma}_2(0)$, then for any $X_0 \in T_{\gamma_1(0)}$,

$$\frac{d}{dt}\Big|_{t=0} P_{0,t}^{\gamma_1}(X_0) = \frac{d}{dt}\Big|_{t=0} P_{0,t}^{\gamma_2}(X_0).$$

and then define linear connection via the formula in proposition above. One can check that the parallel transport of a linear connection satisfies the above four axioms, and conversely a set of parallel transport satisfying the previous four axioms does define a linear connection.

Still let M be a smooth manifold, and ∇ a linear connection on M . Let γ be a loop based at $p \in M$, i.e. $\gamma : [0, 1] \rightarrow M$ is a piecewise smooth curve in M with $\gamma(0) = \gamma(1) = p$. Then we have already seen that the map

$$P^\gamma = P_{0,1}^\gamma : T_p M \rightarrow T_p M$$

is invertible, i.e. $P^\gamma \in GL(T_p M)$. Moreover, if γ_1, γ_2 are two loops based at p , then $\gamma = \gamma_1\gamma_2$ defined by jointing γ_1 and γ_2 is also a loop, and $P^\gamma = P^{\gamma_2} \circ P^{\gamma_1}$. Obviously if we take γ be the constant loop, then P^γ is the identity map in $T_p M$. So the set of all P^γ 's form a subgroup of $GL(T_p M)$.

Definition 1.6. The *holonomy group* of ∇ based at $p \in M$ is

$$\text{Hol}_p(M, \nabla) = \{P^\gamma \mid \gamma \text{ is a loop based at } p\} \subset GL(T_p M).$$

Here are some basic properties of the holonomy groups which we will not prove in this course:

Proposition 1.7. Let ∇ be a linear connection on M , $p, q \in M$, and γ a piecewise smooth curve from p to q .

- (1) $\text{Hol}_p(M, \nabla)$ is a Lie subgroup of $GL(T_p M)$.
- (2) $\text{Hol}_q(M, \nabla) = P^\gamma \text{Hol}_p(M, \nabla) (P^\gamma)^{-1}$.
- (3) If M is simply connected, then $\text{Hol}_p(M, \nabla)$ is connected.

As a first application of parallel transport, we can prove the following result that was used last time:

Proposition 1.8. Let M be a smooth manifold, ∇ be a linear connection whose curvature tensor $R = 0$. Then near any point p , one can find a flat frame, i.e. find a set of vector fields X_1, \dots, X_m on a neighborhood U of p such that

- (1) **[frame]** $\{X_i(q) \mid 1 \leq i \leq m\}$ form a basis of $T_q M$ for every $q \in U$,
- (2) **[flatness]** $\nabla_Y X_i = 0$ for all i and for all vector field Y .

Proof. Without loss of generality, we may take U to be a coordinate neighborhood and $p = (0, \dots, 0)$ the origin. We let $X_i(p) = \partial_i|_p$. Then they form a

basis of T_pM . We extend X^i to the “line” $\{(a, 0, \dots, 0)\}$ by parallelly transporting the vector $X_i(p)$ along the curve $\gamma_0(t) = (t, 0, \dots, 0)$. Then we extend further to the “plane” $\{(a, b, 0, \dots, 0)\}$ by parallelly transporting each $X_i(\gamma_0(a))$ along $\gamma_a(t) = (a, t, 0, \dots, 0)$. Repeat this procedure, we get a set of smooth vector fields $\{X_1, \dots, X_m\}$ on the whole of U . By construction, they are a frame. It remains to prove the flatness.

First by construction, we have

- $\nabla_{\partial_1} X_i = 0$ at any point on the line $(a, 0, \dots, 0)$.
- $\nabla_{\partial_2} X_i = 0$ at any point on the plane $(a, b, 0, \dots, 0)$.

Moreover, since $R = 0$ and $[\partial_1, \partial_2] = 0$, we get

$$\nabla_{\partial_2} \nabla_{\partial_1} X_i = \nabla_{\partial_1} \nabla_{\partial_2} X_i = 0$$

on the plane $(a, b, 0, \dots, 0)$. As a consequence, $\nabla_{\partial_1} X_i$ is parallel along each line $\gamma_a(t) = (a, t, 0, \dots, 0)$, with initial condition $(\nabla_{\partial_1} X_i)(a, 0, \dots, 0) = 0$. By uniqueness, one must have $\nabla_{\partial_1} X_i = 0$ along each γ_a . In other words, we get

- $\nabla_{\partial_1} X_i = 0, \nabla_{\partial_2} X_i = 0$ at any point on the plane $(a, b, 0, \dots, 0)$.

By the same argument, we get

- $\nabla_{\partial_1} X_i = 0, \nabla_{\partial_2} X_i = 0, \nabla_{\partial_3} X_i = 0$ at any point of the form $(a, b, c, 0, \dots, 0)$.

Continue this argument, one can see that $\nabla_{\partial_j} X_i = 0$ for all i, j , at all points in U . As a consequence, X_1, \dots, X_m are flat. \square

Note: in proposition 1.8 don't require M to be a Riemannian manifold!

2. THE PARALLEL TRANSPORT ON RIEMANNIAN MANIFOLDS

Now suppose (M, g) is a Riemannian manifold, and ∇ a metric-compatible linear connection. In other words, for all $X, Y, Z \in \Gamma(TM)$,

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle.$$

So if X, Y are vector fields parallel along a curve γ , then

$$\frac{d}{dt} \langle X(\gamma(t)), Y(\gamma(t)) \rangle = \dot{\gamma}(t) (\langle X, Y \rangle) = \langle \nabla_{\dot{\gamma}(t)} X, Y \rangle + \langle X, \nabla_{\dot{\gamma}(t)} Y \rangle = 0.$$

Thus we get

Lemma 2.1. *If X, Y are both parallel along γ , then $\langle X, Y \rangle$ is a constant on γ .*

As an immediate consequence, we see that if (M, g) is a flat Riemannian manifold, then the flat local frame in proposition 1.8 can be taken to be a flat orthonormal local frame.

For a Riemannian manifold, each tangent space T_pM is not only a linear space, but a linear space with inner product. So for a linear map like P^γ , we would be interested in the question that whether it preserves the inner product structure.

Proposition 2.2. *A linear connection ∇ on M is compatible with g if and only if all of its parallel transports $P_{t_0,t}^\gamma$ are isometries between $T_{\gamma(t_0)}M$ and $T_{\gamma(t)}M$.*

Proof. Let ∇ be a metric-compatible linear connection, γ a curve, and $\{e_i\}$ an orthonormal basis at $\gamma(t_0)$, then

$$\langle P_{t_0,t}^\gamma(e_i), P_{t_0,t}^\gamma(e_j) \rangle = \langle e_i, e_j \rangle.$$

So $P_{t_0,t}^\gamma$ is an isometry.

Conversely, suppose ∇ is a linear connection such that $P_{t_0,t}^\gamma$ are isometries. For any vector fields $X, Y, Z \in \Gamma(TM)$, and any $p \in M$, take a curve γ such that $\gamma(t_0) = p$ and $\dot{\gamma}(t_0) = X_p$. Take an orthonormal basis $\{e_i\}$ at T_pM . Along γ we denote $Y = Y^i(t)e_i(t)$ and $Z = Z^i(t)e_i(t)$. Then

$$\langle Y, Z \rangle = \sum Y^i(t)Z^i(t)$$

along γ . So

$$\nabla_{X_p} \langle Y, Z \rangle = \sum (\nabla_{X_p} Y^i(t)) Z^i(0) + Y^i(0) X_p(Z^i(t)) = \langle \nabla_{X_p} Y, Z_p \rangle + \langle Y_p, \nabla_{X_p} Z \rangle.$$

This completes the proof. \square

As a corollary, we get

Corollary 2.3. *If (M, g) is a Riemannian manifold and ∇ is a linear connection that is compatible with g , then $\text{Hol}_p(M, \nabla) \subset O(T_pM)$.*