LECTURE 12: VARIATIONS AND JACOBI FIELDS

1. Formulas for the first and second variations

Definition 1.1. Let $\gamma : [a, b] \to M$ be a smooth curve, and $\varepsilon > 0$.

(1) A *variation* of γ is a smooth map $f : [a, b] \times (-\varepsilon, \varepsilon) \to M$ so that
$$f(t, 0) = \gamma(t)$$
for all $t \in [a, b]$. In what follows, we will also denote $\gamma_s(t) = f(t, s)$.

(2) A variation f is called *proper* if for every $s \in (-\varepsilon, \varepsilon)$,
$$\gamma_s(a) = \gamma(a) \qquad \text{and} \qquad \gamma_s(b) = \gamma(b).$$

(3) The variation is called a *geodesic variation* if each γ_s is a geodesic.

Let f be a variation of γ. For simplicity we will denote
$$f_s := df(\frac{\partial}{\partial s}), \qquad f_t := df(\frac{\partial}{\partial t}).$$
Note that they are both vector fields near γ.

Lemma 1.2. $\nabla_{f_s} f_t = \nabla_{f_t} f_s$.

Proof.
$$\nabla_{f_s} f_t - \nabla_{f_t} f_s = [f_s, f_t] = df([\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]) = 0.$$
□

By definition,
$$f_t = \dot{\gamma}_s.$$
In particular, on γ one has
$$f_t(\gamma(t)) = \dot{\gamma}.$$

Definition 1.3. We will call
$$V(t) = f_s(\gamma(t))$$
the *variation field* of f along γ. [It is a vector field along γ.]

Last time we calculated the first and second variation of the energy functional
$$E(\gamma) = \frac{1}{2} \int_a^b |\dot{\gamma}(t)|^2 dt$$
with respect to proper variations in local charts, assuming that the whole family of curves sit in one chart. Now we will give a re-formulation of that formula without these assumptions and give an invariant proof.

Theorem 1.4 (The First Variation of Energy)**.** *Let $f(t,s)$ be a smooth variation of a smooth curve γ. Then*

$$\frac{dE(\gamma_s)}{ds} = \langle f_s, f_t\rangle(b,s) - \langle f_s, f_t\rangle(a,s) - \int_a^b \langle f_s, \nabla_{f_t} f_t\rangle dt.$$

As a consequence,

$$\frac{dE(\gamma_s)}{ds}(0) = -\int_a^b \langle V(t), \nabla_{\dot\gamma}\dot\gamma\rangle\, dt - \langle V(a), \dot\gamma(a)\rangle + \langle V(b), \dot\gamma(b)\rangle\,.$$

In particular, if f is also a proper variation, then

$$\frac{dE(\gamma_s)}{ds}(0) = -\int_a^b \langle V(t), \nabla_{\dot\gamma}\dot\gamma\rangle\, dt.$$

Proof. The derivative of $E(\gamma_s)$ is

$$\begin{aligned}
\frac{dE(\gamma_s)}{ds} &= \frac{1}{2}\int_a^b \frac{\partial}{\partial s}\langle \dot\gamma_s(t), \dot\gamma_s(t)\rangle dt \\
&= \frac{1}{2}\int_a^b \nabla_{f_s}\langle f_t, f_t\rangle dt \\
&= \int_a^b \langle \nabla_{f_s} f_t, f_t\rangle dt \\
&= \int_a^b \langle \nabla_{f_t} f_s, f_t\rangle dt \\
&= \int_a^b \frac{\partial}{\partial t}\langle f_s, f_t\rangle dt - \int_a^b \langle f_s, \nabla_{f_t} f_t\rangle dt \\
&= \langle f_s, f_t\rangle(b,s) - \langle f_s, f_t\rangle(a,s) - \int_a^b \langle f_s, \nabla_{f_t} f_t\rangle dt.
\end{aligned}$$

□

Use the same way, one can calculate the first variation of the length. A trick to simplify the computation is the following observation:

$$\frac{\partial}{\partial s}|\dot\gamma_s(t)| = \frac{\partial}{\partial s}\langle f_t, f_t\rangle^{\frac{1}{2}} = \frac{1}{2}\frac{1}{|f_t|}\frac{\partial}{\partial s}\langle f_t, f_t\rangle = \frac{1}{|f_t|}\langle \nabla_{f_t} f_s, f_t\rangle = \langle \nabla_{\dot\gamma(t)} f_s, \frac{f_t}{|f_t|}\rangle.$$

Then following the same computation, one gets

Theorem 1.5 (The First Variation of Length)**.** *Let $f(t,s)$ be a smooth variation of a smooth curve γ. Then*

$$\frac{dL(\gamma_s)}{ds}(0) = -\int_a^b \left\langle V(t), \nabla_{\dot\gamma}\frac{\dot\gamma}{|\dot\gamma|}\right\rangle dt - \left\langle V(a), \frac{\dot\gamma(a)}{|\dot\gamma(a)|}\right\rangle + \left\langle V(b), \frac{\dot\gamma(b)}{|\dot\gamma(b)|}\right\rangle.$$

More generally, one can consider piecewise smooth curves $\gamma : [a, b] \to M$. More precisely, there exists a subdivision

$$a = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = b$$

such that γ is smooth on each interval $[t_i, t_{i+1}]$. We shall consider "piecewise smooth variations" of γ, which are continuous functions $f : [a, b] \times (-\varepsilon, \varepsilon) \to M$ so that f is smooth on each $[t_i, t_{i+1}] \times (-\varepsilon, \varepsilon)$ for each i, and so that f_s is well defined even at t_i's. Applying the previous theorems to each $[t_i, t_{i+1}] \times (-\varepsilon, \varepsilon)$, we get

Corollary 1.6. *Let f be a variation of a piecewise smooth curve γ. Then*

$$\frac{dE(\gamma_s)}{ds}(0) = -\int_a^b \langle V(t), \nabla_{\dot\gamma}\dot\gamma\rangle\, dt - \langle V(a), \dot\gamma(a)\rangle + \langle V(b), \dot\gamma(b)\rangle - \sum_{i=1}^k \langle V(t_i), \dot\gamma(t_i^+) - \dot\gamma(t_i^-)\rangle$$

and

$$\frac{dL(\gamma_s)}{ds}(0) = -\int_a^b \left\langle V(t), \nabla_{\dot\gamma}\frac{\dot\gamma}{|\dot\gamma|}\right\rangle dt - \left\langle V(a), \frac{\dot\gamma(a)}{|\dot\gamma(a)|}\right\rangle + \left\langle V(b), \frac{\dot\gamma(b)}{|\dot\gamma(b)|}\right\rangle - \sum_{i=1}^k \left\langle V(t_i), \frac{\dot\gamma(t_i^+)}{|\dot\gamma(t_i^+)|} - \frac{\dot\gamma(t_i^-)}{|\dot\gamma(t_i^-)|}\right\rangle.$$

Last time we only showed that among smooth curves, geodesics are critical points of the energy functional. A natural question is: If a curve is not smooth, but piecewise smooth, can it be a critical point of the energy functional? Of course for γ be a critical point of the energy functional, it must be a geodesic when restricted to any subinterval where it is smooth, or in other words, it must be "piecewise geodesic".

Corollary 1.7. *If a piecewise smooth curve γ is a critical point of the energy functional, then it is C^1 and thus a geodesic.*

Proof. We can first choose proper variations with variation fields satisfying $V(t_i) = 0$ and deduce that $\nabla_{\dot\gamma}\dot\gamma = 0$ at any smooth point of γ. In particular, the first term in the right hand of the first variation formula vanishes. As a consequence, we have

$$\sum_{i=1}^k \langle V(t_i), \dot\gamma(t_i^+) - \dot\gamma(t_i^-)\rangle = 0$$

for any variation field V. Then for each i we can consider all variation fields so that $V(t_j) = 0$ for all $j \neq i$, and conclude that

$$\langle V(t_i), \dot\gamma(t_i^+) - \dot\gamma(t_i^-)\rangle = 0$$

for any $V(t_i) \in T_{\gamma(t_i)}$. It follows that

$$\dot\gamma(t_i^+) = \dot\gamma(t_i^-),$$

and thus γ is C^1. □

Next we will give an invariant proof for the second variation of energy without restricting ourself to one coordinate chart. As in calculus, the second variation is mainly used near critical points, i.e. near geodesics.

Theorem 1.8 (The Second Variation of Energy)**.** *Let $\gamma : [a,b] \to M$ be a geodesic, and $f(t,s)$ be a smooth variation of γ. Then*

$$\frac{d^2E(\gamma_s)}{ds^2}(0) = -\int_a^b \langle V(t), \nabla_{\dot\gamma}\nabla_{\dot\gamma}V(t) + R(\dot\gamma, V)\dot\gamma(t)\rangle\, dt \\ - \langle V(a), \nabla_{\dot\gamma}V(a)\rangle + \langle V(b), \nabla_{\dot\gamma}V(b)\rangle - \langle \nabla_{V(a)}f_s, \dot\gamma(a)\rangle + \langle \nabla_{V(b)}f_s, \dot\gamma(b)\rangle.$$

In particular, if the variation is proper, then $V(a) = V(b) = 0$, and we have

$$\frac{d^2E(\gamma_s)}{ds^2}(0) = -\int_a^b \langle V(t), \nabla_{\dot\gamma}\nabla_{\dot\gamma}V(t) + R(\dot\gamma, V)\dot\gamma(t)\rangle\, dt.$$

Proof. According to the previous computation,

$$\begin{aligned}\frac{d^2E(\gamma_s)}{ds^2} &= \int_a^b \frac{\partial}{\partial s}\langle \nabla_{f_t}f_s, f_t\rangle dt \\ &= \int_a^b \langle \nabla_{f_s}\nabla_{f_t}f_s, f_t\rangle dt + \int_a^b \langle \nabla_{f_t}f_s, \nabla_{f_s}f_t\rangle dt \\ &= -\int_a^b \langle R(f_s,f_t)f_s, f_t\rangle dt + \int_a^b \langle \nabla_{f_t}\nabla_{f_s}f_s, f_t\rangle dt + \int_a^b \langle \nabla_{f_t}f_s, \nabla_{f_t}f_s\rangle dt \\ &= -\int_a^b \langle R(f_t,f_s)f_t + \nabla_{f_t}\nabla_{f_t}f_s, f_s\rangle dt \\ &\quad + \int_a^b \frac{\partial}{\partial t}\left(\langle \nabla_{f_s}f_s, f_t\rangle + \langle \nabla_{f_t}f_s, f_s\rangle\right) dt - \int_a^b \langle \nabla_{f_s}f_s, \nabla_{f_t}f_t\rangle dt.\end{aligned}$$

Letting $s = 0$, we get the formula we want. □

Remark. Similarly one can write down a formula for the second variation of length, or a formula for "piecewise smooth" variation.

If we take $s = 0$ in the third line of the computation above, we will get the following alternative formula:

$$\frac{d^2E(\gamma_s)}{ds^2}(0) = -\int_a^b \left(\langle V(t), R(\dot\gamma, V)\dot\gamma(t)\rangle - \langle \nabla_{\dot\gamma}V, \nabla_{\dot\gamma}V\rangle\right) dt \\ - \langle \nabla_{V(a)}f_s, \dot\gamma(a)\rangle + \langle \nabla_{V(b)}f_s, \dot\gamma(b)\rangle.$$

As a consequence, we have

Corollary 1.9. *If (M,g) is a Riemannian manifold with non-positive sectional curvature, then any geodesic is locally minimizing.*

2. The Jacobi field

Now suppose γ is a geodesic and f is a geodesic variation of γ, i.e. each

$$\gamma_s = f(\cdot, s)$$

is a geodesic. Let V be its variation field. Then

$$\nabla_{f_t} f_t = \nabla_{\dot{\gamma}_s} \dot{\gamma}_s = 0$$

and

$$[f_t, f_s] = df[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0,$$

so we get

$$\nabla_{f_t} \nabla_{f_t} f_s = \nabla_{f_t} \nabla_{f_s} f_t = -\nabla_{f_s} \nabla_{f_t} f_t + \nabla_{f_t} \nabla_{f_s} f_t + \nabla_{[f_s, f_t]} f_t = R(f_s, f_t) f_t.$$

Taking $s = 0$, we see

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V + R(\dot{\gamma}, V)\dot{\gamma} = 0.$$

Definition 2.1. A vector field X along a geodesic γ is called a *Jacobi field* if

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X + R(\dot{\gamma}, X)\dot{\gamma} = 0.$$

So the variation field of a geodesic variation is a Jacobi field. We will see later that any Jacobi field can be realized as the variation field of some geodesic variation.

Example. Let γ be a geodesic.

(1) Obviously $X = \dot{\gamma}$ is a Jacobi field.
(2) $X = t\dot{\gamma}$ is a Jacobi field since

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} (t\dot{\gamma}) = \nabla_{\dot{\gamma}} [\dot{\gamma} + t \nabla_{\dot{\gamma}} \dot{\gamma}] = 0$$

and

$$R(\dot{\gamma}, t\dot{\gamma})\dot{\gamma} = 0.$$

(3) But $X = t^2 \dot{\gamma}$ is **NOT** a Jacobi field since

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} (t^2 \dot{\gamma}) = \nabla_{\dot{\gamma}} (2t\dot{\gamma}) = 2\dot{\gamma} \neq 0.$$

Theorem 2.2. *Let $\gamma : [a, b] \to M$ be a geodesic, then for any $X_{\gamma(a)}, Y_{\gamma(a)} \in T_{\gamma(a)} M$, there exists a unique Jacobi field X along γ so that*

$$X(a) = X_{\gamma(a)} \qquad \textit{and} \qquad \nabla_{\dot{\gamma}(a)} X = Y_{\gamma(a)}.$$

Proof. Without loss of generality, we may assume that γ is parametrized by arc length. Let $\{e_i(t)\}$ be orthonormal basis at each point $\gamma(t)$ with each $e_i(t)$ parallel along γ, and so that $e_1(t) = \dot{\gamma}(t)$. Note that

$$\nabla_{\dot{\gamma}(t)} e_k(t) = 0$$

for all k. So for a vector field $X = X^i(t) e_i(t)$ along γ,

$$\nabla_{\dot{\gamma}} X = \dot{X}^i(t) e_i(t) \qquad \text{and} \qquad \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X = \ddot{X}^i(t) e_i(t).$$

It follows that the Jacobi field equation becomes

$$\ddot{X}^i(t)e_i(t) + X^i(t){R_{1i1}}^j e_j(t) = 0.$$

This is equivalent to

$$\ddot{X}^i(t) + X^j(t){R_{1j1}}^i = 0, \quad 1 \le i \le m,$$

which is a system of second order linear ODEs. The claim now follows from basic ODE theory. □

Corollary 2.3. *The set of Jacobi fields along γ is a linear space of dimension $2m$, which is canonically ismorphic to $T_{\gamma(a)}M \oplus T_{\gamma(a)}M$.*

Corollary 2.4. *If $X(t)$ is a Jacobi field along γ, and X is not identically zero, then the zeroes of X are discrete.*

Proof. If X has a sequence of zeroes that converges to $\gamma(t_0)$, then $X^i = 0$ for a sequence of points converging to $\gamma(t_0)$. It follows that $X^i(t_0) = 0$ and $\dot{X}^i(t_0) = 0$. In other words, $X(t_0) = 0, \nabla_{\dot\gamma(t_0)}X = 0$. By uniqueness, $X \equiv 0$. □

The obviously Jacobi fields $\dot\gamma$, $t\dot\gamma$ are both *tangent* to γ and are not so interesting for us. We are mainly interested in *normal* Jacobi fields.

Definition 2.5. A Jacobi field along γ is called a *normal Jacobi field* if it is perpendicular to $\dot\gamma$ along γ.

Proposition 2.6. *Let X be a Jacobi field along γ. Then there exists $c^1, d^1 \in \mathbb{R}$ so that*

$$X^\perp = X - c^1 t\dot\gamma - d^1\dot\gamma$$

is a normal Jacobi field along γ.

Proof. $X^\perp$ is a Jacobi field since it is a linear combination of Jacobi fields. According to the Jacobi field equation,

$$\frac{d^2}{dt^2}\langle X, \dot\gamma\rangle = \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}X, \dot\gamma\rangle = -\langle R(\dot\gamma, X)\dot\gamma, \dot\gamma\rangle = 0.$$

It follows that $\langle X, \dot\gamma\rangle$ is a linear function along γ, i.e.

$$\langle X, \dot\gamma\rangle = c_1 t + d_1$$

for some constant $c_1, d_1 \in \mathbb{R}$. Now we let

$$X^\perp = X - c^1 t\dot\gamma - d^1\dot\gamma$$

with $c^1 = \frac{c_1}{|\dot\gamma|^2}, d^1 = \frac{d_1}{|\dot\gamma|^2}$. Then

$$\langle X^\perp, \dot\gamma\rangle = c_1 t + d_1 - c^1 t|\dot\gamma^2| - d^2|\dot\gamma|^2 = 0.$$

□

Corollary 2.7. *A Jacobi field X is normal if and only if*

$$\langle X(a), \dot\gamma(a)\rangle = \langle \nabla_{\dot\gamma(a)} X, \dot\gamma(a)\rangle = 0.$$

In particular, the set of normal Jacobi fields form a linear space of dimension $2m-2$.

Proof. With $X = X^\perp + c^1 t\dot\gamma + d^1\dot\gamma$, we have

$$\langle X(a), \dot\gamma(a)\rangle = (c^1 a + d^1)|\dot\gamma|^2,$$
$$\langle \nabla_{\dot\gamma(a)} X, \dot\gamma(a)\rangle = \langle \nabla_{\dot\gamma(a)}(c^1 t\dot\gamma + d^1\dot\gamma), \dot\gamma(a)\rangle = c^1|\dot\gamma|^2.$$

The conclusion follows. □

Corollary 2.8. *Let X be a Jacobi field so that*

$$\langle X(t_1), \dot\gamma(t_1)\rangle = \langle X(t_2), \dot\gamma(t_2)\rangle = 0$$

for two distinct numbers t_1, t_2. Then X is a normal Jacobi field.

Proof. This follows from the fact that $\langle X, \dot\gamma\rangle$ is a linear function along γ, so it has no more than one zero unless it is identically zero. □

Example. Let (M, g) be a Riemannian manifold with constant sectional curvature k. Then we know

$$R(X,Y)Z = k(\langle X, Z\rangle Y - \langle Y, Z\rangle X).$$

So if γ is a geodesic parametrized by arc-length, then the equation for a *normal* Jacobi field X along γ is

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} X + kX = 0.$$

Now we take an orthonormal basis $\{e_i(t)\}$ of $T_{\gamma(t)}M$ so that

- $e_1(t) = \dot\gamma(t)$,
- each $e_i(t)$ is parallel along γ,

as we did in the proof of theorem 2.2, and let

$$X = \sum_{i=2}^m X^i(t) e_i(t),$$

then the equation for the coefficient $X^i(t)$ is

$$\ddot X^i(t) + kX^i(t) = 0, \qquad 2 \le i \le m.$$

The solution to this equation is

$$X^i(t) = \begin{cases} c^i \frac{\sin(\sqrt{k}t)}{\sqrt{k}} + d^i\cos(\sqrt{k}t), & \text{if } k>0,\\ c^i t + d^i, & \text{if } k = 0,\\ c^i \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}} + d^i \frac{\cosh(\sqrt{-k}t)}{\sqrt{-k}}, & \text{if } k<0,\end{cases}$$

where c^i, d^i are constants.