LECTURE 14: NORMAL COORDINATES

1. The Normal Coordinates

Now we see for any $p \in M$, there exists a neighborhood $U \subset M$ of p and a neighborhood $V \subset T_pM$ of 0 so that the exponential map

$$\exp_p : V \to U$$

is a diffeomorphism. But T_pM, and thus V, is Euclidian, so the triple $\{\exp_p^{-1}, U, V\}$ form a local chart of M near p. Usually one takes such U's to be geodesic balls (and thus V to be Euclidean balls). We will fix an orthonormal basis $\{e_i\}$ of T_pM, which gives us linear coordinates for V. We denote the corresponding coordinate functions on U by $\{x^i\}$.

Definition 1.1. The local chart $\{U; x^1, \cdots, x^m\}$ described above is called *normal coordinate system* at p.

This is a particularly nice coordinate system. For example, the parametric equation for the geodesic $\gamma(t) = \exp_p(tX_p)$, where $X_p = x^i\partial_i$, is given by

$$\gamma : x(t) = (tx^1, tx^2, \cdots, tx^m),$$

Moreover, we have

Lemma 1.2. *Let $\{U; x^1, \cdots, x^m\}$ be a normal coordinate system at p. Then*

(1) *For all $1 \le i, j \le m$, $g_{ij}(p) = \delta_{ij}$.*
(2) *For all $1 \le i, j, k \le m$, $\Gamma_{ij}^k(p) = 0$.*
(3) *For all $1 \le i, j, k \le m$, $\partial_k g_{ij}(p) = 0$.*

Proof. (1) This is obvious, since $\partial_i|_p = d(\exp_p)_0 e_i = e_i$.

(2) Put the geodesic parametric equation

$$x(t) = (tx^1, tx^2, \cdots, tx^m).$$

into the geodesic equation, we get for $1 \le k \le m$,

$$0 = \ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma_{ij}^k(\gamma(t)) = x^i x^j \Gamma_{ij}^k(\gamma(t)).$$

Letting $t = 0$, we conclude that for any $1 \le k \le m$,

$$x^i x^j \Gamma_{ij}^k(p) = 0$$

for all x^i, x^j. It follows that for any fixed k, all eigenvalues of the symmetric matrix $(\Gamma_{ij}^k)_{1 \le i,j \le m}$ are 0. So $\Gamma_{ij}^k(p) = 0$ for all i, j, k.

(3) This is a consequence of (2) and the metric compatibility. ∎

From (1) and (3) above, we see that the Taylor's expansion of $g_{ij}(x)$ at p using normal coordinates start with the constant term is δ_{ij}, and has no linear term. What is the next term?

Theorem 1.3. *With respect to normal coordinates near p, the functions g_{ij}'s admit the following Taylor expansion at $x = 0$,*

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}R_{iklj}(p)x^k x^l + O(|x^3|). \tag{1}$$

Proof. Let $(U; x^1, \cdots, x^m)$ be a normal coordinate system near p. Fix x^i's and let γ be the geodesic starting at p in the direction $X_p = x^i\partial_i$,

$$\gamma(t) = (tx^1, \cdots, tx^m), \quad 0 \le t \le 1.$$

For each $1 \le i \le m$, consider a geodesic variation

$$f_i(t,s) = (tx^1, \cdots, t(x^i + s), \cdots, tx^m).$$

Its variation field $V_i = t\partial_i$ is thus a Jacobi field along γ, which satisfies

$$V_i(0) = 0, \quad \nabla_{\dot\gamma(0)} V_i = \partial_i.$$

In view of the first equation and the Jacobi field equation

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}V + R(\dot\gamma, V)\dot\gamma = 0$$

we also have

$$\nabla^{(2)}_{\dot\gamma(0)} V_i := \nabla_{\dot\gamma(0)}\nabla_{\dot\gamma} V_i = 0.$$

Moreover, if we take the $(k-2)^{th}$ covariant derivative of the Jacobi field equation, we get

$$\nabla^{(k)}_{\dot\gamma} V_i + \sum_{l=0}^{k-2} \binom{k-2}{l} (\nabla^{(k-2-l)}_{\dot\gamma} R)(\dot\gamma, \nabla^{(l)}_{\dot\gamma} V_i)\dot\gamma = 0,$$

where we used the facts

$$\nabla_W(R(X,Y)Z) = (\nabla_W R)(X,Y)Z + R(\nabla_W X, Y)Z + R(X, \nabla_W Y)Z + R(X,Y)\nabla_W Z$$

and $\nabla_{\dot\gamma}\dot\gamma = 0$. Taking $k = 3$, we get

$$\nabla^{(3)}_{\dot\gamma} V_i + (\nabla_{\dot\gamma} R)(\dot\gamma, V_i)\dot\gamma + R(\dot\gamma, \nabla_{\dot\gamma} V_i)\dot\gamma = 0.$$

Evaluating at $t = 0$, and using $V_i(0) = 0$, we get

$$\nabla^{(3)}_{\dot\gamma(0)} V_i = -R(X_p, \partial_i)X_p = R(\partial_i, X_p)X_p.$$

So if we let

$$h(t) = t^2 g_{ij}(tx^1, \cdots, tx^m) = \langle V_i(t), V_j(t)\rangle,$$

then

$$\begin{aligned}
h(0) &= \langle V_i(0), V_j(0)\rangle = 0,\\
h'(0) &= \langle \nabla_{\dot\gamma(0)} V_i, V_j(0)\rangle + \langle V_i, \nabla_{\dot\gamma(0)} V_j(0)\rangle = 0,\\
h''(0) &= \langle \nabla^{(2)}_{\dot\gamma(0)} V_i, V_j(0)\rangle + 2\langle \nabla_{\dot\gamma(0)} V_i, \nabla_{\dot\gamma(0)} V_j\rangle + \langle V_i, \nabla^{(2)}_{\dot\gamma(0)} V_j(0)\rangle = 2\delta_{ij},\\
h'''(0) &= \sum_{l=0}^{3} \binom{3}{l} \langle \nabla^{(3-l)}_{\dot\gamma(0)} V_i, \nabla^{(l)}_{\dot\gamma(0)} V_j\rangle = 0,\\
h''''(0) &= \sum_{l=0}^{4} \binom{4}{l} \langle \nabla^{(4-l)}_{\dot\gamma(0)} V_i, \nabla^{(l)}_{\dot\gamma(0)} V_j\rangle = 8R(\partial_i, X_p, X_p, \partial_j).
\end{aligned}$$

As a consequence, we get

$$\begin{aligned}
g_{ij}(tx^1, \cdots, tx^m) &= \frac{1}{t^2}\langle V_i(t), V_j(t)\rangle\\
&= \frac{1}{t^2}\left(\delta_{ij}t^2 + \frac{8}{4!}R(\partial_i, X_p, X_p, \partial_j)t^4 + O(t^5)\right)\\
&= \delta_{ij} + \frac{1}{3}R(\partial_i, X_p, X_p, \partial_j)t^2 + O(t^3)\\
&= \delta_{ij} + \frac{1}{3}R(\partial_i, \partial_k, \partial_l, \partial_j)(tx^k)(tx^l) + O(t^3).
\end{aligned}$$

This proves the theorem. □

Remark. One can continue to calculate $\nabla^{(k)}_{\dot\gamma(0)} V_i$'s and get a full expansion of g_{ij} in normal coordinates. For example, the next two terms are

$$\frac{1}{6}R_{iklj;r}x^k x^l x^r + (\frac{1}{20}R_{iklj;rs} + \frac{2}{45}R_{kil}{}^{m}R_{rjsm})x^k x^l x^r x^s.$$

Taking derivative of (1), we get

$$\partial_r g_{ij} = \frac{1}{3}R_{irlj}x^l + \frac{1}{3}R_{ikrj}x^k + O(|x|^2).$$

Taking derivative again and evaluate at p, we get

$$\partial_s\partial_r g_{ij}(0) = \frac{1}{3}R_{irsj}(p) + \frac{1}{3}R_{isrj}(p).$$

As a consequence, we get

Corollary 1.4. *With respect to normal coordinates, one has*

$$R_{ijkl}(p) = \frac{1}{2}(\partial_i\partial_l g_{jk} + \partial_j\partial_k g_{il} - \partial_i\partial_k g_{jl} - \partial_j\partial_l g_{ik})(0).$$

Proof. The right hand side equals

$$\frac{1}{6}(R_{jlik} + R_{jilk} + R_{ikjl} + R_{ijkl} - R_{jkil} - R_{jikl} - R_{iljk} - R_{ijlk})(p),$$

which equals $R_{ijkl}(p)$ by using symmetries of the Riemann curvature tensor. $\square$

Remark. This is Riemann's original definition of the curvature tensor.

2. Geometric Meaning of Curvatures

Now we are ready to give geometric interpretations of sectional, Ricci and scalar curvatures.

Theorem 2.1. *Let $\Pi_p \subset T_pM$ be a 2-dimensional plane. Denote by C_r^0 the circle of radius r in Π_p centered at p, and $C_r = \exp_p(C_r^0)$. Let L_r be the length of C_r. Then*

$$\lim_{r\to 0}\frac{2\pi r - L_r}{r^3} = \frac{\pi}{3}K(\Pi_p).$$

Proof. Take an orthonormal basis $\{e_1, \cdots, e_m\}$ of T_pM so that Π_p is spanned by e_1, e_2, and consider the normal coordinate system with respect to $\{e_i\}$. Then for r small, the cricel C_r has equation

$$C_r : x^1(t) = r\cos t,\ x^2(t) = r\sin t,\ x^k(t) = 0\ (k \geq 3).$$

It follows that

$$\begin{aligned}|\dot{C}_r(t)|^2 &= g_{ij}(C_r(t))\dot{x}^i(t)\dot{x}^j(t)\\ &= \left(1+\frac{1}{3}R_{1221}x^2x^2\right)\dot{x}^1\dot{x}^1 + \left(1+\frac{1}{3}R_{2112}x^1x^1\right)\dot{x}^2\dot{x}^2 + 2\frac{1}{3}R_{1212}x^1x^2\dot{x}^1\dot{x}^2 + O(r^5)\\ &= r^2 - \frac{r^4}{3}K(\Pi_p) + O(r^5).\end{aligned}$$

So

$$\begin{aligned}L_r = \mathrm{Length}(C_r) &= \int_0^{2\pi}|\dot{C}_r|dt = r\int_0^{2\pi}\sqrt{1-\frac{r^2}{3}K(\Pi_p)+O(r^3)}dt\\ &= 2\pi r - \frac{\pi}{3}K(\Pi_p)r^3 + O(r^4).\end{aligned}$$

This implies the theorem. $\square$

So the sectional curvature measures the deviation of the geodesic circle to the standard circle in Euclidean space.

To give a geometric interpretation of the Ricci curvature, we first prove

Lemma 2.2. *In a normal coordinate system near p, we have*

$$\det(g_{ij}) = 1 - \frac{1}{3}Ric_{kl}(p)x^kx^l + O(|x|^3).$$

Proof. Let $A = \ln(g_{ij})$. Since $(g_{ij}) = I + \left(\frac{1}{3}R_{iklj}(p)x^kx^l + O(|x^3|)\right)$, and

$$\ln(I+B) = B + \frac{B^2}{2} + \cdots + \frac{B^k}{k} + \cdots,$$

we see

$$A = \left(\frac{1}{3}R_{iklj}(p)x^k x^l + O(|x^3|)\right).$$

In particular,

$$\mathrm{tr}(A) = \frac{1}{3}R_{ikli}(p)x^k x^l + O(|x|^3) = -\frac{1}{3}Ric_{kl}(p)x^k x^l + O(|x|^3),$$

where in the last step we used the fact $\delta^{ij} = g^{ij}$. As a consequence,

$$\det(g_{ij}) = \det(\exp(A)) = \exp(\mathrm{tr}(A)) = 1 - \frac{1}{3}R_{kl}(p)x^k x^l + O(|x|^3).$$

□

As an immediate consequence, we get

Corollary 2.3. *The volume element $\sqrt{\det(g_{ij})}$ in normal coordinates has the expansion*

$$\sqrt{\det(g_{ij})} = 1 - \frac{1}{6}Ric_{kl}(p)x^k x^l + O(|x|^3).$$

In particular, we get

Corollary 2.4. *Let $u_p \in S_pM$ be a unit tangent vector at p, and let $\gamma(t)$ be the geodesic starting at p with $\dot\gamma(0) = u_p$. Then*

$$\sqrt{\det(g_{ij}(\gamma(t)))} = 1 - \frac{\mathrm{Ric}(u_p)}{6}t^2 + O(t^3).$$

Proof. Take an orthonormal basis of T_pM so that $u_p = e_1$. Then with respect to the corresponding normal coordinates, the geodesic $\gamma(t) = \exp_p(tu_p)$ can be written as

$$\gamma : x^1(t) = t, \quad x^2 = \cdots = x^m = 0.$$

It follows

$$\sqrt{\det(g_{ij}(\gamma(t)))} = 1 - \frac{1}{6}Ric_{11}(p)t^2 + O(t^3) = 1 - \frac{1}{3}\mathrm{Ric}(u_p)t^2 + O(t^3).$$

□

So the Ricci curvature measures the change of the volume element in the given direction.

Finally let's study the scalar curvature S. We have

Theorem 2.5. *For r small enough,*

$$\mathrm{Vol}(B_r(p)) = \omega_m r^m \left(1 - \frac{S(p)}{6(m+2)}r^2 + O(|r^3|)\right),$$

where ω_m is the volume of the unit ball in $\mathbb{R}^m$.

Proof. By definition

$$\begin{aligned}\mathrm{Vol}(B_r(p)) &= \int_{B_r(0)} \sqrt{\det(g_{ij})}dx^1 \cdots dx^m \\ &= \int_{B_r(0)} \left(1 - \frac{1}{6}Ric_{kl}(p)x^k x^l\right) dx^1 \cdots dx^m + O(r^3) \\ &= \omega_m r^m - \frac{Ric_{kl}(p)}{6} \int_{B_r(0)} x^k x^l dx^1 \cdots dx^m + O(r^3).\end{aligned}$$

An elementary computation shows

$$\int_{B_r(0)} x^k x^l dx^1 \cdots dx^m = \frac{\omega_m}{m+2} r^{m+2} \delta^{kl}$$

and the theorem follows. □

So the scalar curvature measures the deviation of the volume of a geodesic ball to the standard ball.

Corollary 2.6. *The surface area of geodesic sphere $S_r(p)$ is*

$$\mathrm{Area}(S_r(p)) = m\omega_m r^{m-1} - \frac{S(p)}{6}\omega_m r^{m+1} + O(r^{m+2}).$$