# LECTURE 15: COMPLETENESS AND CONVEXITY

## 1. The Hopf-Rinow Theorem

Recall that a Riemannian manifold $(M, g)$ is called *geodesically complete* if the maximal defining interval of any geodesic is $\mathbb{R}$. On the other hand, any Riemannian manifold $(M, g)$ admits a *metric structure* given by

$$d(p, q) = \inf\{L(\gamma) \mid \gamma \text{ is a piecewise smooth curve connecting } p \text{ to } q\},$$

and thus we can talk about the completeness of $d$: a metric space is *complete* if any Cauchy sequence in it converges. The following theorem says that for Riemannian manifolds, the two notions of completeness coincide.

**Theorem 1.1** (Hopf-Rinow)**.** *Let $(M, g)$ be a connected Riemannian manifold. Then the following statements are equivalent:*

(1) *$(M, d)$ is a complete metric space.*
(2) *$(M, g)$ is geodesically complete.*
(3) *There exists $p \in M$ so that $\exp_p$ is defined for all $X_p \in T_pM$.*
(4) [Heine-Borel property] *Any bounded closed subset in $M$ is compact.*

*Moreover, each of the previous statements implies*

(5) *There exists $p \in M$ so that any point $q \in M$ can be connected to $p$ by a minimal geodesic, i.e. a geodesic of length $d(p, q)$.*

*Proof.* We shall prove

$$(4) \Longrightarrow (1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (5) \quad \text{and} \quad (3) + (5) \Longrightarrow (4).$$

$\boxed{(4)\Rightarrow(1)}$ This is a standard result in general topology.

$\boxed{(1)\Rightarrow(2)}$ Let $\gamma$ be any normal geodesic on $M$. By the existence and uniqueness theorem, the maximal defining interval of $\gamma$ must be an open interval $(a, b)$. If $b < \infty$, then we can take a sequence $s_i \to b-$. In particular, $\{s_i\}$ is a Cauchy sequence in $\mathbb{R}$. But $\gamma$ is a normal geodesic, so

$$d(\gamma(s_i), \gamma(s_j)) \le |s_i - s_j|.$$

As a consequence, $\{\gamma(s_i)\}$ is a Cauchy sequence in $(M, d)$. If follows that there exists a $p \in M$ so that $\gamma(s_i) \to p$.

Since $\mathcal{E}$ is open and $(p, 0) \in \mathcal{E}$, there exists $\varepsilon > 0$ so that $(q, Y_q) \in \mathcal{E}$ for any $q$ with $d(q, p) < \varepsilon$ and any $Y_q \in T_qM$ with $|Y_q| < 2\varepsilon$. So if we take $i$ large enough so that $b - s_i < \frac{\varepsilon}{2}$ and thus $d(\gamma(s_i), p) < \frac{\varepsilon}{2}$, then $\gamma(t; \gamma(s_i), \varepsilon\dot{\gamma}(s_i))$ is defined for

$t \in [0,1]$. In other words, the geodesic $\gamma_1(t) = \gamma(t; \gamma(s_i), \dot\gamma(s_i))$ is well defined for $0 < t < \varepsilon$. Since $\gamma_1$ coincides with $\gamma$ at $s_i$, they must be *the same*. In particular, $\gamma$ can be defined for all $t < s_i + \varepsilon$, which exceeds the upper bound $b$, a contradiction.

Similarly by considering the "reverse geodesic" one can see that $a = -\infty$. So any normal geodesic on $M$, and thus any geodesic on $M$, has defining interval $\mathbb{R}$.

(2)⇒(3) This is obvious. ($(M, g)$ is geodesically complete $\Leftrightarrow \mathcal{E} = TM$. )

(3)⇒(5) Denote $r = d(p, q)$. (Here we used the connectedness!) We have already seen that there exists $0 < \delta < r$ so that the exponential map exp is a diffeomorphism from $B_\delta(0) \in T_pM$ to $B_\delta(p) \in M$. Note that $S_\delta(p) = \exp_p(S_\delta(0))$ is compact. Since the distance function is continuous (we proved this in lecture 2), there exists $p_0 \in S_\delta(p)$ so that

$$d(p_0, q) = \inf_{p' \in S_\delta(p)} d(p', q).$$

Let $\gamma$ be the normal geodesic from $p$ to $p_0$. By (3), $\gamma$ is defined over $\mathbb{R}$. We define

$$A = \{s \in [\delta, r] \mid d(\gamma(s), q) = r - s\}.$$

We will show $\sup A = r$, which implies $\gamma(r) = q$.

To prove this, we first notice that $\delta \in A$, since

$$r = d(p, q) = \inf_{p' \in S_\delta(p)} (d(p, p') + d(p', q)) = \delta + \inf_{p' \in S_\delta(p)} d(p', q) = \delta + d(\gamma(\delta), q).$$

So $A$ is nonempty.

Secondly, it's easy to see that $A$ is closed, since the function

$$f(s) = d(\gamma(s), q) - r + s$$

is continuous and that $A = f^{-1}(0) \cap [\delta, r]$.

Now let $s_0 = \sup A$. Since $A$ is nonempty and closed, $s_0 \in A$. Suppose $s_0 < r$. Then by repeating the previous argument, we know that there exists $0 < \delta' < r - s_0$ and $p_0' \in S_{\delta'}(\gamma(s_0))$ so that

$$d(p_0', q) = \min_{p' \in S_{\delta'}(\gamma(s_0))} d(p', q) = d(\gamma(s_0), q) - \delta'.$$

Since $s_0 \in A$, we get

$$d(p_0', q) = r - s_0 - \delta'.$$

So by triangle inequality,

$$d(p_0', p) \geq d(p, q) - d(p_0', q) = r - (r - s_0 - \delta') = s_0 + \delta'.$$

On the other hand, the curve $\tilde\gamma$ by connecting $p$ to $\gamma(s_0)$ along $\gamma$ and then connecting $\gamma(s_0)$ to $p_0'$ by <u>*the*</u> minimal geodesic has length exactly $s_0 + \delta'$. So $\tilde\gamma$, with the arc-length parametrization, must be a geodesic. Obviously $\tilde\gamma$ has to coincide with $\gamma$. In other words, $p_0' = \gamma(s_0 + \delta')$. As a consequence,

$$d(\gamma(s_0 + \delta'), q) = r - (s_0 + \delta'),$$

i.e. $s_0 + \delta' \in A$. This conflicts with the fact that $s_0 = \sup A$.

(3)+(5)⇒(4) Let $K \subset M$ be a bounded closed set. Then there exists a constant $C > 0$ so that $d(p,k) < C$ for all $k \in K$. According to (3) and (5), $K \subset \exp_p(\overline{B_C(0)})$, where $\overline{B_C(0)}$ is the *closed* ball of radius $C$ in $T_pM$, which is compact in $T_pM$. Since $\exp_p$ is smooth, $\exp_p(B_C(0))$ is also compact. Thus $K$, as a closed subset of a compact set, is compact.

□

**Definition 1.2.** A Riemannian manifold $(M, g)$ satisfying any of (1)-(4) is called a *complete Riemannian manifold.*

*Remarks.*

- Condition (5) is NOT enough to guarantee that $(M, g)$ is complete. For example, the open unit ball $B_1(0)$ in $(\mathbb{R}^n, g_0)$ satisfies (5), which is not complete.
- For a general metric space, condition (1) does NOT imply condition (4). For example, one can consider a countable infinite set $\{x_i \mid i \in \mathbb{N}\}$ and define a metric on it via $d(x_i, x_j) = 1$ for all $i \neq j$. In this space, the only Cauchy sequences are eventually-constant sequences which of course converge. However, the whole space is closed and bounded but not compact. So as metric spaces, Riemannian manifolds are special (and *nice*) metric spaces.

**Corollary 1.3.** *Any two points in a connected complete Riemannian manifold can be connected by a minimal geodesic.*

**Corollary 1.4.** *If $(M, g)$ is complete and connected, then for any $p \in M$, $\exp_p : T_pM \to M$ is surjective.*

**Corollary 1.5.** *Any compact Riemannian manifold is complete.*

## 2. Convex Neighborhoods

By definition, for any $r < \text{inj}(M, p)$, the exponential map

$$\exp_p : B_r(0) \subset T_pM \to B_r(p) \subset M$$

is a diffeomorphism. Moreover, for any $q \in B_r(p)$, there is a unique geodesic connecting $q$ to the center $p$ whose length is less than $r$. Of course such a geodesic is the unique minimizing geodesic connecting $q$ to $p$.

**Theorem 2.1.** *For any point $p \in M$ there exists a neighborhood $W$ of $p$ and a number $\delta > 0$ so that for each $q \in W$, one has* $\text{inj}(M, q) \geq \delta$ *and $B_\delta(q) \supset W$.*

*Remark.* As a consequence, any two points in $W$ can be connected by a unique minimizing geodesic. The neighborhood satisfying theorem 2.1 is called a *totally normal neighborhood.*

*Proof.* We consider the map

$$F : \mathcal{E} \to M \times M, \quad F(q, X_q) = (q, \exp_q X_q).$$

Then $F(p, 0) = (p, p)$ and

$$dF_{(p,0)} = \begin{pmatrix} I & * \\ 0 & I \end{pmatrix}.$$

By the inverse function theorem, $F$ is a local diffeomorphism. In other words, $F$ maps a neighborhood $\mathcal{U}$ of $(p, 0)$ diffeomorphically onto a neighborhood $\mathcal{W}$ of $(p, p)$ in $M \times M$. By shrinking $\mathcal{U}$ if necessary, one can take $\mathcal{U}$ to be of the form

$$\mathcal{U} = \{(q, X_q) \mid q \in U, |X_q| < \delta\},$$

where $U$ is a small neighborhood of $p$ in $M$. Now choose a neighborhood $W$ of $p$ in $M$ so that $W \times W \subset \mathcal{W}$. Obviously $\delta$ and $W$ obtained by this way satisfies the assertion of the theorem. □

Now let $W$ be a totally normal neighborhood, so that any two points in $W$ can be joint by a unique minimizing geodesic. A natural question is: Does these minimizing geodesic all lie in $W$? Obviously to answer this question, we need to require $W$ to satisfy some kind of "convexity".

Recall that a region in $\mathbb{R}^n$ is convex if any two points in the region can be connected by a line segment that lies in the region.

**Definition 2.2.** A subset $S \in M$ is called *strongly convex*, or *geodesically convex*, if for any $p, q \in S$ there is a unique normal minimal geodesic $\gamma$ joining $p$ to $q$, and $\gamma$ is contained in $S$.

*Example.* On $(S^2, g_{S^2})$, any geodesic ball of radius $r < \frac{\pi}{2}$ is strongly convex.

*Remark.* Obviously each strongly convex set is contractible, and the intersection of a family of strongly convex sets in $M$ is again strongly convex if it is not empty. This fact is used in Algebraic Topology to product *good covers* on arbitrary manifolds, which makes Čech cohomology much simpler to understand.

**Theorem 2.3** (Whitehead)**.** *For any $p \in M$ there exists $\rho > 0$ so that the geodesic ball $B_\rho(p)$ is strongly convex.*

In proving the theorem, we will need

**Lemma 2.4.** *For any $p \in M$, there exists $\eta > 0$ so that for any $0 < r < \eta$ and any $q \in S_r(p)$, any geodesic $\gamma$ that is tangent to $S_r(p)$ at $q$ stays out of $B_r(p)$ for some neighborhood of $q$.*

*Remark.* A geodesic ball satisfying the assertion of the previous lemma is called locally convex. Recall that a region in $\mathbb{R}^n$ is convex if for any point in the boundary of the region, any tangent line of the boundary surface at that point outside the region. In other words, in $\mathbb{R}^n$ locally convex is the same as convex. This fact does not holds for general Riemannian manifolds. For example, for the standard cylinder

$S^1 \times \mathbb{R}$, a geodesic ball of radius $\frac{\pi}{2} < r < \pi$ is locally convex but is not strongly convex.

*Proof of Whitehead Theorem.* Take $\eta$ as in Lemma 2.4 and take $W$ and $\delta < \frac{\eta}{2}$ as in Theorem 2.1. Take $\rho < \delta$ so that $B_\rho(p) \subset W$. We claim that $B_\rho(p)$ is strongly convex.

In fact, since $B_\rho(p) \subset W$, any two points $q_1, q_2 \in B_\rho(p)$ can be connected by a minimal geodesic of length no more than $\delta$. It follows that $\mathrm{Im}(\gamma) \subset B_\eta(p)$ since $\rho + \delta < \eta$. If the interior of $\gamma$ is not totally contained in $B_\rho(p)$, then there is a point $q_3$ on $\gamma$ so that

$$\rho < \sup_{q' \in \mathrm{Im}\gamma} d(p, q') := d(p, q_3) := \xi < \eta.$$

It is clear that $\gamma$ is tangent to $S_\xi(p)$ and lies totally in $B_\xi(p)$. This contradict with Lemma 2.4. □

*Proof of Lemma 2.4.* Let $W$ be a totally normal neighborhood of $p$. For any $q \in W$ and any $Y_q \in T_qM$ with $|Y_q| = 1$, let $u(t; q, Y_q) = \exp_p^{-1}(\gamma(t; q, Y_q))$ (this is defined for $|t|$ small) and let

$$F(t; q, Y_q) = |u(t; q, Y_q)|^2.$$

It is clear that $u$, and thus $F$, are smooth, and

$$\frac{\partial F}{\partial t} = 2\left\langle u, \frac{\partial u}{\partial t}\right\rangle, \qquad \frac{\partial^2 F}{\partial t^2} = 2\left\langle u(t), \frac{\partial^2 u}{\partial t^2}\right\rangle + 2\left\langle \frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right\rangle.$$

Observe that for $q = p$ and for any $Y_q = Y_p \in S_pM$, we have $u(t; p, Y_p) = tY_p$ and hence

$$\left.\frac{\partial^2 F}{\partial t^2}\right|_{p, X_p} = 2\langle Y_p, Y_p\rangle = 2.$$

By continuity, there exists a neighborhood $V \subset W$ of $p$ so that for any $q \in V$ and any $Y_q \in S_qM$, $\frac{\partial^2 F}{\partial t^2}(0; q, Y_q) > 0$. Take $\eta > 0$ so that $B_\eta(p) \subset V$. We claim that this $\eta$ satisfies the assertion of the lemma.

In fact, for any $r < \eta$, let $\gamma(t; q, Y_q)$ be a normal geodesic tangent to $S_r(p)$ at $q = \gamma(0; q, Y_q) = \exp_p(u(0; q, Y_q))$. Then the tangent vector of $\gamma(t; q, Y_q) = \exp_p(u(t; q, Y_q))$ at $q$ is $(d\exp_x)_{u(0;q,Y_q)} \frac{\partial u}{\partial t}(0; q, Y_q)$, which should be a tangent vector of $S_r(p)$ at $q$. According to the Gauss Lemma, $\frac{\partial u}{\partial t}(0; q, Y_q)$ is a tangent vector of $S_r(0) \in T_pM$ at the point $u(0; q, Y_q)$. It follows

$$\frac{\partial F}{\partial t}(0; q, Y_q) = 2\left\langle u(0; q, Y_q), \frac{\partial u}{\partial t}(0; q, Y_q)\right\rangle = 0.$$

As a consequence, for $q \in B_\eta(p)$ and $Y_q \in S_qM$, the function $F(t; q, Y_q)$ takes its strict local minimum at $t = 0$, where its value is $F(0; q, Y_q) = r^2$. So for $|t|$ small enough, we must have $F(t; q, Y_q) > r^2$, which proves the lemma. □