LECTURE 16: CONJUGATE AND CUT POINTS

1. Conjugate Points

Let (M,g) be Riemannian and $\gamma : [a,b] \to M$ a geodesic. Then by definition,

$$\exp_p((t-a)\dot\gamma(a)) = \gamma(t).$$

We know that $\exp_p$ is a diffeomorphism near 0. But it may fail to be a diffeomorphism away from 0.

Definition 1.1. We say $q = \gamma(t_0)$ $(t_0 > a)$ is *conjugate* to $p = \gamma(a)$ along γ if $\exp_p$ is singular at $(t_0-a)\dot\gamma(a)$, i.e. $(d\exp_p)_{(t_0-a)\dot\gamma(a)}$ is not of full rank.

It turns out that conjugate points are closely related to Jacobi fields:

Theorem 1.2. *Let $\gamma : [a,b] \to M$ is a geodesic. Then*

1. *$q = \gamma(t_0)$ is a conjugate point of $p = \gamma(a)$ if and only if there exists a nonvanishing Jacobi field X along γ so that $X(a) = 0$ and $X(t_0) = 0$.*
2. *Moreover, the dimension of all Jacobi fields vanishing at both p and q equals $\dim\ker(d\exp_p)_{(t_0-a)\dot\gamma(a)}$.*

Proof. In the proof of theorem 1.7 in lecture 13 (i.e. any Jacobi field is the variation field of some geodesic variation), we showed that if X is a Jacobi field along γ with

$$X(a) = 0 \quad \text{and} \quad Y_{\gamma(a)} = \nabla_{\dot\gamma(a)}X,$$

then it is the variation field of

$$f(t,s) = \exp_{\gamma(a)}((t-a)(\dot\gamma(a) + sY_{\gamma(a)})).$$

In particular we have

$$X(t) = (d\exp_{\gamma(a)})_{(t-a)\dot\gamma(a)}((t-a)Y_{\gamma(a)}).$$

It follows

$$\begin{aligned}
&q = \gamma(t_0) \text{ is conjugate to } p = \gamma(a)\\
\Longleftrightarrow\ &\ker(d\exp_p)_{(t_0-a)\dot\gamma(a)} \neq 0\\
\overset{(*)}{\Longleftrightarrow}\ &Y_{\gamma(a)} \neq 0 \text{ and } 0 = X(t_0) = (d\exp_{\gamma(a)})_{(t_0-a)\dot\gamma(a)}((t_0-a)Y_{\gamma(a)})\\
\Longleftrightarrow\ &\text{There is a nonzero Jacobi field } X \text{ along } \gamma \text{ so that } X(a) = 0 \text{ and } X(t_0) = 0.
\end{aligned}$$

This proves the first assertion. The second assertion also follows from $(*)$, together with the canonical isomorphism $\mathcal{J} \simeq T_pM \oplus T_pM$. $\square$

Definition 1.3. If $q = \gamma(t_0)$ is a conjugate point of p along γ, we call

$$\dim \ker(d \exp_p)_{(t_0-a)\dot{\gamma}(a)}$$

the *multiplicity* of the conjugate point q.

Remarks.

(1) If q is conjugate to p along a geodesic γ, then p is conjugate to q along a suitable chosen geodesic (the *reverse* of γ starting at q), with the same multiplicity.

(2)Any Jacobi field satisfying $X(a) = 0$ and $X(t_0) = 0$ is a normal Jacobi field. [c.f. corollary 2.7 in lecture 12]

(3) Let $\dim M = m$. Recall (Lecture 12)

- The set $\mathcal{J}$ of all Jacobi fields $\simeq T_pM \oplus T_pM$, and thus the set of Jacobi fields with $X(a) = 0$ is a linear space of dimension m.
- There is no tangent Jacobi fields (i.e. of the form $a\dot{\gamma} + bt\dot{\gamma}$) with $X(a) = 0$ and $X(t_0) = 0$.

So the set of normal Jacobi fields with $X(a) = 0$ is a linear space of dimension $m-1$. It follows that the multiplicity of a conjugate point is no more than $m - 1$.

Example. Let $M = S^m$ be the round sphere whose sectional curvature is 1. Let γ be a normal geodesic starting from any p. Then according to lecture 12, any normal Jacobi field along γ with $X(0) = 0$ must be of the form

$$X(t) = \sum_{i=2}^{m} c^i \sin(t) e_i(t),$$

where $\{e_i(t)\}$ is an orthonormal basis at each $\gamma(t)$, with $e_1(t) = \dot{\gamma}(t)$. It follows that if γ has length less than π, then there is no conjugate point of p, and if the length of γ is between π and 2π, then the "anti-podal" point $\gamma(\pi) = \bar{p}$ is the only conjugate point to the north pole along any geodesic starting at p, and its multiplicity equals $m - 1$.

Example. Similar computation shows that if M has constant sectional curvature $k \leq 0$, then to any point there is no conjugate point. In particular, on cylinder or torus there is no conjugate point. In fact the same conclusion holds for any Riemannian manifold whose sectional curvatures are non-positive (maybe not constant).

Proposition 1.4. *Suppose $q = \gamma(t_0)$ is NOT conjugate to $p = \gamma(a)$ along γ. Then for any $X_p \in T_pM$ and $X_q \in T_qM$, there exists a unique Jacobi field X along γ so that $X(a) = X_p$ and $X(t_0) = X_p$.*

Proof. Let $\mathcal{J}$ be the set of all Jacobi fields along γ. Define a mapping

$$\Theta : \mathcal{J} \to T_pM \times T_qM, \quad X \mapsto \Theta(X) = (X(a), X(t_0)).$$

Since q is not a conjugate point of p, Θ is injective. But Θ is linear, and $\dim \mathcal{J} = \dim(T_pM \times T_qM) = 2m$ are of same dimension, we conclude that Θ is an linear isomorphism, which proves the theorem. □

We have already seen any *local* (i.e. short) geodesic is minimizing among nearby curves. The following theorem claims that a *global* geodesic is minimizing among nearby curves if and only if it contains no pair of conjugate points.

Theorem 1.5 (Jacobi)**.** *Let $\gamma : [a, b] \to M$ be a geodesic. Denote $p = \gamma(a), q = \gamma(b)$.*

(1) *If there is no conjugate points of p along γ, then there exists $\varepsilon > 0$ so that for any piecewise smooth curve $\bar{\gamma} : [a, b] \to M$ from p to q satisfying $\mathrm{dist}(\gamma(t), \overline{\gamma}(t)) < \varepsilon$, we have $L(\bar{\gamma}) \geq L(\gamma)$, with equality hold if and only if $\bar{\gamma}$ is a reparametrization of γ.*
(2) *If there exists $\bar{t} \in (a, b)$ so that $\bar{q} = \gamma(\bar{t})$ is a conjugate point of p, then there is a proper variation of γ so that $L(\gamma_s) < L(\gamma)$ for $0 < |s| < \varepsilon$.*

Remark. In part (1) we only claim that γ is minimizing among nearby curves. It is possible that there exists other shorter geodesics. For example, one can look at cylinders, in which case there is no conjugate point (since the sectional curvature is 0). Between any two points there are infinitely many geodesics, each is minimizing among nearby curves, but only one of them is globally minimizing.

We will prove the theorem next time.

2. The Cut Locus

Let $\gamma : [a, b] \to M$ be a geodesic in M, $p = \gamma(a)$ and $q = \gamma(b)$ the end points. We have already seen the question "whether γ is shortest" is a subtle question:

- If q is very close to p, γ is the shortest curve connecting p to q.
- γ will never be shortest after the first conjugate point of p along γ.
- Before the first conjugate point of p, γ is shortest among nearby curves connecting p to q.
- Even if p has no conjugate point along γ, it's still possible that γ is not shortest curve connecting p to q.

Definition 2.1. Let (M, g) be a complete Riemannian manifold, $p \in M$ a point, and $\gamma : [0, \infty) \to M$ a normal geodesic with $\gamma(0) = p$. If

$$t_0 := \sup\{t \mid \gamma([0, t]) \text{ is a minimizing geodesic}\} < +\infty,$$

then we will call $\gamma(t_0)$ the *cut point* of p along γ. We will denote by $\mathrm{Cut}(p)$ the set of all cut points of p along all geodesics that start from p, and call it the *cut locus* of p.

Remark. If M is compact, then $\mathrm{Cut}(p) \neq \varnothing$ for all p.

Example. On $\mathbb{R}^m$ and $\mathbb{H}^m$ (each endowed with the canonical metric), there exists only one normal minimal geodesic joining any two given points. So $\mathrm{Cut}(p) = \varnothing$ for all p.

Example. For S^m with the round metric, $\mathrm{Cut}(p) = \{\bar{p}\}$ for any $p \in M$, where $\bar{p} = -p$ is the antipodal point of p. Note that $\bar{p}$ is also the first conjugate point of p.

Example. For the cylinder $S^1 \times \mathbb{R}$ endowed with the canonical metric, if $p = (e^{i\theta_0}, z_0)$, then $\mathrm{Cut}(p) = \{(e^{i(\theta_0+\pi)}, z) \mid z \in \mathbb{R}\}$ is the vertical line "opposite to p". Note that p has no conjugate points at all.

The following theorem relates cut points with conjugate points:

Theorem 2.2. *Suppose $\gamma(t_0)$ is the cut point of $p = \gamma(0)$ along a normal geodesic γ, then at least one of the following assertion holds:*

(1) $\gamma(t_0)$ *is the first conjugate point of p along γ.*

(2) $\gamma(t_0)$ *is the first point along γ so that there exists another normal geodesic $\sigma \neq \gamma$ from p to $\gamma(t_0)$ with length $L(\sigma) = t_0 = L(\gamma|_{[0,t_0]})$.*

Proof. Take a decreasing sequence $t_i \to t_0^+$. Let σ_i be a normal minimizing geodesic connecting p to $\gamma(t_i)$. Then $L(\sigma_i) < t_i$. Note that $\{\dot{\sigma}_i(0)\}$ is a sequence in the unit sphere S_pM. By passing to a subsequence, we may assume $\dot{\sigma}_i(0) \to X \in S_pM$. Let σ be the normal geodesic with $\sigma(0) = p$, $\dot{\sigma}(0) = X$. Then by continuity, σ is a minimizing geodesic connecting p to $\gamma(t_0)$, thus $L(\sigma) = t_0$.

Case 1: $X = \dot{\gamma}(0)$. Let $s_i = L(\sigma_i)$. Then by definition of cut point, $s_i < t_i$. It follows that $s_i\dot{\sigma}_i(0) \neq t_i\dot{\gamma}(0)$. But

$$\exp_p(s_i\dot{\sigma}_i(0)) = \sigma_i(s_i) = \gamma(t_i) = \exp_p(t_i\dot{\gamma}(0)),$$

so $\exp_p$ is not a local diffeomorphism near $t_0\dot{\gamma}(0)$. In other words, $\exp_p$ is singular at $t_0\dot{\gamma}(0)$. So $\gamma(t_0)$ is a conjugate point of p. Obviously it has to be the first conjugate point, otherwise $\gamma([0, t_0])$ is not minimizing.

Case 2: $X \neq \dot{\gamma}(0)$. Then σ is a geodesic that is different from γ. We have

$$t_0 = L(\gamma|_{[0,t_0]}) \leq L(\sigma) = \lim_i L(\sigma_i) \leq \lim_i t_i = t_0.$$

So $L(\sigma) = t_0$. To show that $\gamma(t_0)$ is the first point along γ with this property, we argue by contradiction. If there exists a $\bar{t} < t_0$ and a normal geodesic $\bar{\sigma}$ connecting p to $\gamma(\bar{t})$ so that $L(\bar{\sigma}) = \bar{t}$, then the curve $\bar{\gamma}$ defined by connecting $\bar{\sigma}$ with $\gamma|_{[\bar{t},t_0]}$ is a piecewise smooth but not smooth curve connecting p to $\gamma(t_0)$ whose length is t_0. But according to the first variation formula, any piecewise smooth but not smooth curve is not a minimizing curve. We claim that $\gamma|_{[0,t_0]}$ is also not a minimizing curve, since it has the same length as $\bar{\gamma}$. This contradicts with the definition of cut point. □

Corollary 2.3. *If $q \in \mathrm{Cut}(p)$, then $p \in \mathrm{Cut}(q)$.*

Proof. If q is the cut point of p along γ, then γ is minimizing between p and q. It follows that the "opposite geodesic" $-\gamma$ is also miniming between q and p. Moreover, by the theorem above, either q is the first conjuge point of p along γ, or there exists a different normal geodesic σ joint p to q which has length $L(\sigma) = \mathrm{dist}(p, q)$. In both cases $-\gamma$ is no longer minimizing after p. So $p \in \mathrm{Cut}(p)$. □

Corollary 2.4. *If $q \notin \mathrm{Cut}(p)$, then there exists a unique minimizing geodesic joining p to q.*

Proof. If there exist two minimizing geodesics γ, σ joining p to q, then γ is minimizing between p and q, and is no longer minimizing after q. So $q \in \mathrm{Cut}(p)$. □

Remark. One can show that the function $f : SM \to \mathbb{R} \cup \{\infty\}$ defined by

$$f(p, X_p) = \begin{cases} t_0, & \text{if } \gamma_{p,X_p}(t_0) \text{ is the cut point of } p \text{ along } \gamma, \\ +\infty, & \text{if } p \text{ has no cut point along } \gamma_{p,X_p}. \end{cases}$$

is a continuous function. It follows that $\mathrm{Cut}(p)$ is a closed subset of measure zero in M.

3. Smoothness of Distance Function

Now let's fix $p \in M$ and consider the distance function

$$d_p : M \to \mathbb{R}, \quad d_p(q) = \mathrm{dist}(p, q).$$

As we have already seen, d_p is a continuous function. However, it is not hard to see that $d_p \in C^\infty(M)$. In fact, d_p is never smooth at p.

Example: Consider (S^2, g_{S^2}). Let $\bar{p} = -p$ be the antipodal point of p. Then for q near $\bar{p}$, $d_p(q) = \pi - d_{\bar{p}}(q)$. It follows that d_p is also not smooth at $\bar{p}$.

Theorem 3.1. *The function d_p is smooth on $M \setminus \mathrm{Cut}(p) \cup \{p\}$. Moreover, for each $q \in M \setminus \mathrm{Cut}(p) \cup \{p\}$, if we let γ^q be the unique normal minimizing geodesic from p to q, then the gradient of d_p at q is*

$$(\nabla d_p)(q) = \dot{\gamma}^q(d_p(q)).$$

Proof. For each $q \in M \setminus \mathrm{Cut}(p) \cup \{p\}$, let γ^q be the unique normal minimizing geodesic from p to q and denote $X^q = \dot{\gamma}^q(0) \in S_pM$. Let

$$A = \{L(\gamma^q)X^q \mid q \in M \setminus \mathrm{Cut}(p) \cup \{p\}\}.$$

Then $A \subset T_pM \setminus \{0\}$ is an open set and $\exp_p : A \to M \setminus \mathrm{Cut}(p) \cup \{p\}$ is smooth. Moreover, at each vector in A, $\exp_p$ is nonsingular and thus a local diffeomorphism. Since $\exp_p$ is globally one-to-one on A, it is a diffeomorphism from A to $M \setminus \mathrm{Cut}(p) \cup \{p\}$. It follows that $\exp_p^{-1} : M \setminus \mathrm{Cut}(p) \cup \{p\} \to A \subset T_pM \setminus \{0\}]$ is smooth. Thus $d_p(q) = |\exp_p^{-1}(q)|$ is smooth on $M \setminus \mathrm{Cut}(p) \cup \{p\}$.

To calculate its gradient at q, we choose any $X \in T_qM$ and let $\sigma(s)$ be a smooth curve in $M \setminus \mathrm{Cut}(p) \cup \{p\}$ tangent to X at $q = \sigma(0)$. Now we consider the variation of γ^q so that γ^q_s be the unique minimizing geodesic from p to $\sigma(s)$. Observe that the variation field vector of this variation at the point q is exactly X. So according to the first variation formula,

$$X(d_p) = \left.\frac{d}{ds}\right|_{s=0} d_p(\sigma(s)) = \left.\frac{d}{ds}\right|_{s=0} L(\gamma^q_s) = \langle X, \dot{\gamma}^q(d_p(q))\rangle.$$

It follows that $(\nabla d_p)(q) = \dot{\gamma}^q(d_p(q))$. $\square$

Remarks. (1) One can show that if there exists two minimizing geodesic from p to q, then d_p is not differentiable at q.

(2) By using the second variation formula one can calculate the Hessian of d_p on $M \setminus \mathrm{Cut}(p) \cup \{p\}$.