LECTURE 18: THE THEOREMS OF AMBROSE AND CARTAN-HADAMARD

1. The Theorem of Ambrose

Let M, N be smooth manifolds. Recall

Definition 1.1. A smooth map $f : M \to N$ is said to be a *smooth covering map* if for any $q \in N$, there is a neighborhood V of q in N and disjoint open subsets U_α of M so that $f^{-1}(V) = \cup_\alpha U_\alpha$, and for each α, $f : U_\alpha \to V$ is a diffeomorphism.

Remarks. (1) If f is a covering map, then $\dim M = \dim N$ and f is surjective.
(2) If N is simply connected, then f is a global diffeomorphism.

Now let (M, g) and (N, h) be Riemannian manifolds of the same dimension.

Definition 1.2. A smooth map $f : M \to N$ is a *local isometry* if $g = f^*h$.

Note that a local isometry f don't have to be a global diffeomorphism. However, for all $p \in M$ and all $X_p, Y_p \in T_pM$,

$$g(X_p, Y_p) = h(df_p(X_p), df_p(Y_p)).$$

In other words, for any p, the map

$$df_p : (T_pM, g_p) \to (T_{f(p)}N, h_{f(p)})$$

is a linear isometry. As consequences, we see

1. f is a local diffeomorphism everywhere.
2. For any curve $\gamma \subset M$, one has $|\dot\gamma|_{\gamma(t)} = |df_{\gamma(t)}(\dot\gamma)|_{f(\gamma(t))}$, which implies
$$L(\gamma) = L(f(\gamma))$$
3. M and N has *the same* Riemannian metric at corresponding points, and thus the same Levi-Civita connection, the same Riemannian curvature. In particular, f maps geodesics into geodesics.

We shall prove the following theorem, which states that the geometric property of f ("local isometry") implies topological property of f ("covering map"). It is easy to see that the theroem fails if we don't assume (M, g) to be complete.

Theorem 1.3 (Ambrose)**.** *Let (M, g) and (N, h) be Riemannian manifold, and $f : M \to N$ a local isometry. Suppose (M, g) is complete, then f is a smooth covering map, and (N, h) is complete.*

Proof. We will prove this in four steps.
Step 1 Lift geodesics in N to geodesics in M:

Lemma 1.4. *Under the conditions of the theorem, given any geodesic* $\gamma : [a,b] \to N$ *and any* $p \in f^{-1}(\gamma(a))$*, we can "lift"* γ *to a geodesic* $\bar{\gamma} : [a,b] \to M$ *so that* $\gamma(t) = f(\bar{\gamma}(t))$ *and* $\bar{\gamma}(a) = p$*. Moreover, the lift is unique.*

Proof. Since $df_p : T_pM \to T_{\gamma(a)}N$ is a linear isometry, one can find a unique $X_p \in T_pM$ so that

$$df_p(X_p) = \dot{\gamma}(a).$$

Let $\bar{\gamma} : [a,b] \to M$ be the geodesic in M with

$$\bar{\gamma}(a) = p, \dot{\bar{\gamma}}(a) = X_p.$$

(We used completeness here!) Then $f \circ \bar{\gamma}$ is a geodesic in N with same initial conditions as γ. We conclude that $\gamma = f \circ \bar{\gamma}$.
The uniqueness of lift also follows directly from the fact that $df_p : T_pM \to T_{\gamma(a)}N$ is a linear isometry. $\square$

Step 2 f is surjective.

Since any local isometry is a local diffeomorphism which must be an open map, $f(M)$ is open in N. If f is not surjective, then $f(M)$ is not closed, i.e. there exists $\bar{q} \in \overline{f(M)} \setminus f(M)$. Take δ small enough so that the geodesic ball $B_\delta(\bar{q})$ is a normal neighborhood of $\bar{q}$, i.e. any point in $B_\delta(q)$ can be connected to $\bar{q}$ by a unique minimal geodesic. Now take any $q \in B_\delta(\bar{q}) \cap f(M)$, then there exists $p \in M$ so that $f(p) = q$. If we denote by $\gamma : [0,1] \to N$ the minimal geodesic in N with

$$\gamma(0) = q, \gamma(1) = \bar{q}.$$

According to the lemma above, there exists a geodesic $\bar{\gamma} : [0,1] \to M$ in M so that

$$f \circ \bar{\gamma} = \gamma.$$

It follows that

$$\bar{q} = \gamma(1) = f(\bar{\gamma}(1)).$$

This contradicts with the choice of $\bar{q}$.
Step 3 (N,h) is complete.

Let $\gamma : [a,b] \to N$ is an arbitrary geodesic in N. Since f is surjective, there exists $p \in f^{-1}(\gamma(a))$. By the lemma in step 1, we can lift γ to a geodesic $\bar{\gamma}$ through p. Since (M,g) is complete, $\bar{\gamma}$ is defined for all t. It follows that $\gamma = f \circ \bar{\gamma}$ is defined for all t. So $(N,\hbar)$ is complete.
Step 4 Verify covering properties.

Fix any $q \in N$, we may assume

$$f^{-1}(q) = \{p_\alpha\}_{\alpha \in I}.$$

We take δ small enough so that $V = B_\delta(q) \subset N$ is a normal geodesic ball around q. For each α, we let

$$U_\alpha = B_\delta(p_\alpha) \subset M.$$

We note that each point in U_α can be connected to p_α through a unique minimizing geodesic of length less than δ. In particular, p_α has no conjugate point in U_α, thus U_α is also a normal geodesic ball around p_α. It remains to prove

(1) $f^{-1}(V) = \cup_\alpha U_\alpha$.
(2) $f : U_\alpha \to V$ is diffeomorphism.
(3) For $\alpha \neq \beta$, $U_\alpha \cap U_\beta = \emptyset$.

Step 4.1 Proof of (1): For any $p \in f^{-1}(V)$, one has $f(p) \in V$. We let $\gamma : [a, b] \to N$ be the minimal geodesic in V connecting $f(p)$ to q, and $\bar{\gamma}$ its lift starting at p. Then

$$f(\bar{\gamma}(b)) = \gamma(b) = q,$$

so there exists α so that $\bar{\gamma}(b) = p_\alpha$. Note $L(\bar{\gamma}) = L(\gamma) < \delta$, we conclude that $p \in U_\alpha$. So $f^{-1}(V) \subset \cup_\alpha U_\alpha$.

Conversely, for any point $p \in U_\alpha$, there is a minimal geodesic $\bar{\gamma} : [0, 1] \to M$ connecting p_α to p with length $< \delta$. It follows that $\gamma = f \circ \bar{\gamma}$ is a geodesic starting from q with length $< \delta$. So

$$f(p) = f(\bar{\gamma}(1)) = \gamma(1) \in V,$$

and thus $U_\alpha \subset f^{-1}(V)$.

Step 4.2 Proof of (2): Since local isometry maps geodesics into geodesics, and geodesics are determined by initial values, we have

$$\exp_q [df_{p_\alpha}(X_\alpha)] = f(\exp_{p_\alpha}(X_\alpha))$$

for any $X_\alpha \in T_{p_\alpha}M$. Moreover, when restricted to balls of radius δ, both $\exp_q$ and $\exp_{p_\alpha}$ are diffeomorphisms. Since df is an linear isomorphism which is also a diffeomorphism on the whole $T_{p_\alpha}M$, we conclude that

$$f = \exp_q \circ df_{p_\alpha} \circ \exp_{p_\alpha}^{-1}$$

when restricted to balls of radius δ, so in particular $f : U_\alpha \to V$ is a diffeomorphism.

Step 4.3 Proof of (3): Suppose there exists $p \in U_\alpha \cap U_\beta$ and $\alpha \neq \beta$. Let $\bar{\gamma}_\alpha$ and $\bar{\gamma}_\beta$ be the minimal geodesic from p to p_α and p_β respectively. Then $f(\bar{\gamma}_\alpha)$ and $f(\bar{\gamma}_\beta)$ are minimal geodesics in N, both from $f(p)$ to q. It follows that

$$f(\bar{\gamma}_\alpha) = f(\bar{\gamma}_\beta),$$

and both $\bar{\gamma}_\alpha$ and $\bar{\gamma}_\beta$ are lifts of $f(\bar{\gamma}_\alpha)$ from p. By uniqueness of lift, $\bar{\gamma}_\alpha = \bar{\gamma}_\beta$ and thus $p_\alpha = p_\beta$. $\square$

2. The Theorem of Cartan-Hadamard

Now we are ready to prove

Theorem 2.1 (Cartan-Hadamard)**.** *Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature, then for any $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a covering map. In particular, if M is also simply connected, then $\exp_p$ is a diffeomorphism (and thus M is non-compact).*

Before we prove the Cartan-Hadamard theorem, we need the following lemma saying that any non-positive curvature manifold has no conjugate point. (Recall we showed this for non-positive constant curvature case by explicitly computing the Jacobi fields.)

Lemma 2.2. *Let (M, g) be as above. Then for any $p \in M$, p has no conjugate point along any geodesic emanating from p.*

Proof. Let γ be a geodesic emanating from p and X be a normal Jacobi field along γ with $X(0) = 0$. Let

$$f(t) = \langle X(t), X(t)\rangle.$$

Then

$$f'(t) = 2\langle \nabla_{\dot\gamma(t)} X, X\rangle$$

and thus

$$f''(t) = 2\langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} X, X\rangle + 2|\nabla_{\dot\gamma} X|^2 = -2R(\dot\gamma, X, \dot\gamma, X) + 2|\nabla_{\dot\gamma} X|^2 \geq 0.$$

Since $f'(0) = 0$, we must have $f'(t) \geq 0$ for all $t > 0$, i.e. f is non-decreasing. But we also know that for t small enough, $f(t) > 0$ since the zeroes of a Jacobi field is discrete. If follows that $f(t) > 0$ for all t. In other words, X has no zero along γ. So p has no conjugate point along γ. □

Proof of the Cartan-Hadamard Theorem. According to lemma 2.2,

$$\exp : T_pM \to M$$

is a local diffeomorphism everywhere. Let $\bar g = (\exp_p)^* g$, then $\bar g$ is a Riemannian metric on T_pM such that

$$\exp_p : (T_pM, \bar g) \to (M, g)$$

is a local isometry. Note that the geodesics in $(T_pM, \bar g)$ passing $0 \in T_pM$ are exactly the straight lines passing 0, which are defined for all t. It follows that $\exp_0$ is defined for all $X_0 \in T_0(T_pM)$. According to the Hopf-Rinow theorem, $(T_pM, \bar g)$ is complete. It follows from Ambrose Theorem that $\exp_p : T_pM \to M$ is a covering map.

If M is simply connected, then any covering map to M must be a global homeomorphism. Since $\exp_p$ is also a local diffeomorphism, it must be global diffeomorphism. □

Corollary 2.3. *Let (M, g) be a complete simply connected flat manifold. Then (M, g) is isometric to $(\mathbb{R}^n, g_0)$.*

Proof. Choose any p. Identify T_pM with $\mathbb{R}^n$ and let $\bar{g} = \exp_p^* g$ on $\mathbb{R}^n$. We have already proved that the map

$$\exp_p : (\mathbb{R}^n, \bar{g}) \to (M, g)$$

is both a diffeomorphism and a local isometry. So it is a global isometry. Since g is flat, $\bar{g}$ is then a flat metric on $\mathbb{R}^n$. But two flat metrics on $\mathbb{R}^n$ differ only by a linear isomorphism [Prove this!]. So the conclusion follows. □

Definition 2.4. A complete simply-connected Riemannian manifold with non-positive curvature is called a *Cartan-Hadamard manifold*, or an *Hadamard manifold*.

Definition 2.5. Let (M, g) be a complete Riemannian manifold. We say $p \in M$ is a *pole* of (M, g) if $\exp_p : T_pM \to M$ is non-singular everywhere.

By repeating the proof of Cartan-Hadamard theorem word by word, we can prove

Theorem 2.6. *If p is a pole of (M, g), then $\exp_p : T_pM \to M$ is a smooth covering.*