LECTURE 19: THE THEOREMS OF BONNET-MYERS, SYNGE AND PREISSMAN

1. Bonnet-Myers Theorem

Now let't turn to Riemannian manifolds with positive curvature.

Theorem 1.1 (Bonnet-Myers). *Let (M,g) be a complete Riemannian manifold whose Ricci curvature satisfies*

$$\mathrm{Ric}(X_p) \ge (m-1)\kappa$$

for all $X_p \in SM$, where κ is a positive constant independent of X_p. Then M is compact, and its diameter is bounded by

$$\mathrm{diam}(M) := \sup_{p,q\in M} \mathrm{dist}(p,q) \le \frac{\pi}{\sqrt{\kappa}}.$$

Proof. For any $p, q \in M$, let $\gamma : [0,1] \to M$ be a minimal geodesic joining p to q. Obviously it's enough to show $L(\gamma) \le \frac{\pi}{\sqrt{\kappa}}$. Suppose on the contrary that

$$L(\gamma) = l > \frac{\pi}{\sqrt{\kappa}}.$$

Let $\{e_i(t)\}$ be parallel vector fields along γ which form an orthonormal basis at each point $\gamma(t)$ and so that $e_1(t) = \frac{\dot\gamma(t)}{l}$. For $j = 2, \cdots, m$, we define

$$V_j(t) = \sin(\pi t) e_j(t).$$

Then $V_j(0) = V_j(1) = 0$, and

$$I(V_j, V_j) = -\int_0^1 \langle V_j, \nabla_{\dot\gamma}\nabla_{\dot\gamma} V_j + R(\dot\gamma, V_j)\dot\gamma\rangle dt = \int_0^1 \sin^2(\pi t)(\pi^2 - l^2 R(e_1, e_j, e_1, e_j))dt.$$

Summing over j, we get

$$\sum_{j=2}^m I(V_j, V_j) = \int_0^1 \sin^2(\pi t)((m-1)\pi^2 - l^2 \mathrm{Ric}(e_1))dt < 0.$$

So there exists some $j \ge 2$ so that

$$I(V_j, V_j) < 0.$$

If follows that there exists $\bar q = \gamma(t_0)$ with $0 < t_0 < 1$ which is conjugate to p along γ. In particular, γ is not shortest. A contradiction. □

Remarks. (1) One cannot weaken the condition on Ricci curvature to $Ric > 0$ or even $K > 0$. For example, consider the paraboloid

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}.$$

It is a surface of revolution with $K > 0$, which is not compact.

(2) The estimate is optimal in the following sense: Let M be the standard sphere of radius $\frac{1}{\sqrt{\kappa}}$, then it has Ricci curvature $(m-1)\kappa$ and diameter $\frac{\pi}{\sqrt{\kappa}}$. (Note: the diameter here is not the standard diameter as a subset in $\mathbb{R}^n$.)

(3) We will prove the following result of S. Y. Cheng later: If (M, g) satisfies the conditions of the Bonnet-Myers theorem and $\text{diam}(M) = \frac{\pi}{\sqrt{\kappa}}$, then (M, g) is isometric to the standard sphere of radius $\frac{1}{\sqrt{\kappa}}$.

Corollary 1.2. *Let (M, g) be a complete Riemannian manifold whose Ricci curvature is bounded below by a positive number. Then $\pi_1(M)$ is finite.*

Proof. Let $\overline{M}$ be the universal covering of M, endowed with the pull-back metric $\bar{g} = \pi^* g$. Then $(\overline{M}, \bar{g})$ is also a complete Riemannian manifold whose Ricci curvature is bounded below by a positive number. By Bonnet-Myers theorem, $\overline{M}$ is compact. As a consequence, $\pi : \overline{M} \to M$ has to be a finite covering. So $\pi_1(M)$ is finite. □

It particular, we see that if M, N are compact, $\pi_1(M)$ is infinite, then $M \times N$ admits no Riemannian metric of positive Ricci curvature.

2. Synge's Theorem

Another application of the second variation formula to Riemannian manifolds with positive curvature is

Theorem 2.1 (Synge)**.** *Let (M, g) be a compact Riemannian manifold with positive sectional curvature.*

1. *If M is even dimensional and orientable, then M is simply connected.*
2. *If M is odd dimensional, then M is orientable.*

Before proving the theorem, we need two lemmas:

Lemma 2.2. *Let (M, g) be a compact Riemannian manifold. Then for any nontrivial free homotopy class $\mathcal{C}$, there exists a closed geodesic γ whose length is minimal in $\mathcal{C}$.*

Proof. As an exercise. (See more details in PSet 3.) □

Lemma 2.3. *Let (M, g) be an orientable Riemannian manifold, and $\gamma : [a, b] \to M$ be a closed curve, i.e. $\gamma(a) = \gamma(b) := p$. Then the parallel transport $P^{\gamma}_{a,b} : T_pM \to T_pM$ has determinant 1.*

Proof. In lecture 10 we have already seen that $P^\gamma_{a,b} \in O(T_pM)$. So it is enough to show $\det P^\gamma_{a,b} > 0$. To prove this, we take a *positive* m-form ω on M, and let $\{e_i\}$ be a *positive* basis of T_pM, i.e.

$$\omega(e_1, \cdots, e_m) > 0.$$

Let $e_j(t) = P^\gamma_{a,t}(e_j)$ be the parallel transport of $\{e_i\}$ along γ. Then

$$\omega(e_1(t), \cdots, e_m(t)) \neq 0$$

for all t. It follows that

$$\omega(e_1(b), \cdots, e_m(b)) > 0.$$

But

$$\omega(e_1(b), \cdots, e_m(b)) = (\det P^\gamma_{a,b})\omega(e_1, \cdots, e_m),$$

so we must have $\det P^\gamma_{a,b} > 0$. □

Proof of Synge's Theorem. (1) Suppose M is not simply connected. Then there exists a nontrivial closed geodesic $\gamma : [0,1] \to M$ which is minimum in its free homotopy class. Since the parallel transport $P^\gamma_{0,1} \in SO(T_pM)$ and satisfies

$$P^\gamma_{0,1}(\dot\gamma(0)) = \dot\gamma(0),$$

we can find $X_p \in E_p$ such that

$$P^\gamma_{0,1}(X_p) = X_p,$$

where E_p is the orthogonal complement of $\dot\gamma(0)$ in T_pM. (Here, we used the condition that $\dim M$ is even, so that $\dim E$ is odd!)

Now let $X(t)$ be the parallel vector field along γ with $X(0) = X_p$. Then

$$X(1) = P^\gamma_{0,1}(X_p) = X_p.$$

Thus for the variation γ_s of γ whose variation field is X, we have

$$\frac{d^2}{dt^2}\Big|_{t=0} E(\gamma_s) = -\int_0^1 \langle X, \nabla_{\dot\gamma}\nabla_{\dot\gamma}X + R(\dot\gamma, X)\dot\gamma\rangle dt = -\int_0^1 R(\dot\gamma, X, \dot\gamma, X\rangle dt < 0.$$

This contradicts with the fact that γ is minimum in its homotopy class.

(2) Suppose M is not orientable, then there is a nontrivial free homotopy class $\mathcal{C}$ so that for any closed curve $\gamma : [0,1] \to M$ in $\mathcal{C}$, $\det P^\gamma_{0,1} = -1$. [Prove this!] We will take γ to be the one with minimal length in this class. Since $P^\gamma_{0,1}(\dot\gamma(0)) = \dot\gamma(0)$, we see

$$\det P^\gamma_{0,1}|_E = -1,$$

where $E = (\dot\gamma(0))^\perp$ is the orthogonal complement of $\dot\gamma(0)$ in T_pM. Since E is even dimensional, again we conclude that there exists $X_p \in E$ so that

$$P^\gamma_{0,1}(X_p) = X_p.$$

Now by the same argument of the proof of Synge theorem we conclude that γ is not minimum in its homotopy class, a contradiction. □

Corollary 2.4. *If (M, g) is a compact even dimensional Riemannian manifold of positive sectional curvature, and M is not orientable, then $\pi_1(M) = \mathbb{Z}_2$.*

Proof. Let $\overline{M}$ be the orientable double covering of M, endowed with the induced pull-back metric. Then $\overline{M}$ is orientable and satisfies all the conditions in Synge theorem. It follows that $\overline{M}$ is simply connected and thus $\pi_1(M) = \mathbb{Z}_2$. □

As a consequence, $\mathbb{RP}^2 \times \mathbb{RP}^2$ admits no metric of positive sectional curvature since $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2) = (\mathbb{Z}/2)^2$. Recall that it is still unknown whether $S^2 \times S^2$ admits a positive sectional curvature metric: that is the Hopf's conjecture.

Remark. In the odd dimensional case we cannot say too much of its fundamental group. In fact, it is well-known that S^{2n+1} can be the universal covering space of a lot of spaces of constant curvature 1.

3. Preissman's theorem

We can also study the fundamental group of negative curved manifolds.

Theorem 3.1 (Preissman)**.** *Let (M, g) be a compact Riemannian manifold with negative sectional curvature, and let $\{1\} \neq H \subset \pi_1(M)$ be a nontrivial abelian subgroup of the fundamental group. Then H is infinite cyclic.*

Remarks. (1) Recall: a cyclic group is a group generated by one element.

(2) An an immediate consequence, we see that manifolds like T^m, $\mathbb{RP}^m$ admits no metric of negative sectional curvature.

(3) The theorem was strengthened by Byers to: Under the same assumption, any nontrivial solvable subgroup of $\pi_1(M)$ is infinite cyclic.

(4) For any closed surface M_g of genus $g \geq 2$, there is Riemannian metric of constant negative sectional curvature. [We mentioned this in lecture 9]. The fundamental group of M_g is

$$\langle a_1, b_1, \cdots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_1 b_1 a_1^{-1} b_1^{-1} = e \rangle.$$

This group is not abelian, while all its abelian subgroups are isomorphic to $\mathbb{Z}$.

Before we prove the theorem, we need some preparations.

Definition 3.2. Let (M, g) be a complete simply-connected Riemannian manifold, and $\gamma : \mathbb{R} \to M$ a geodesic. An isometry $f : M \to M$ is called a *translation* along γ if f has no fixed point, and $f(\gamma) = \gamma$.

Let (M, g) be any complete Riemannian manifold and $\pi : \widetilde{M} \to M$ be the universal covering. We endow with $\widetilde{M}$ the pull back metric $\pi^* g$. Recall that for each element $\alpha \in \pi_1(M)$, one can define a deck transformation $f_\alpha : \widetilde{M} \to \widetilde{M}$ as follows: for each $\tilde{p} \in \widetilde{M}$, there is a loop γ based at $p = \pi(\tilde{p})$ whose homotopy class is α. Let $\tilde{\gamma}$ be the lift of γ with starting point $\tilde{p}$. Define $f_\alpha(\tilde{p})$ be the endpoint of $\tilde{\gamma}$.

One can prove that f_α is well-defined, is an isometry, and $f_\beta \circ f_\alpha = f_{\beta\alpha}$. Moreover, f_α has no fixed point if $\alpha \neq e$. Let Γ be the group of all deck transformations, then it is isomorphic to $\pi_1(M)$.

Now suppose $0 \neq \alpha \in \pi_1(M)$, and let γ be a minimal closed geodesic in the homotopy class α. Let $\tilde{\gamma}$ be a lift of γ to $\widetilde{M}$.

Lemma 3.3. *$f_\alpha : (\widetilde{M}, \tilde{g}) \to (\widetilde{M}, \tilde{g})$ is a translation along $\tilde{\gamma}$.*

Proof. Exerciese. □

Another lemma that is needed in the proof is

Lemma 3.4. *Let (M, g) be a complete simply connected Riemannian manifold of non-positive sectional curvature. Consider the geodesic triangle with vertices $p_1, p_2, p_3 \in M$. Let a, b, c be the lengths of sides and A, B, C be the corresponding opposite angles. Then*

(1) $a^2 + b^2 - 2ab\cos C \leq c^2$.
(2) $A + B + C \leq \pi$.

Further more, if the sectional curvature is negative, then these inequalities are strict.

Proof. We will prove this later. □

As a consequence of this lemma, we have

Corollary 3.5. *Suppose (M, g) has negative sectional curvature, then any translation $f : \widetilde{M} \to \widetilde{M}$ fixes only one geodesic.*

Proof. Suppose there are two geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $\widetilde{M}$ such that $f(\tilde{\gamma}_i) = \tilde{\gamma}_i$. First we claim that $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = \emptyset$. Otherwise there are at least two points in $\tilde{\gamma}_1 \cap \tilde{\gamma}_2$ (since f has no fixed point). This contradicts with the fact that $\exp_{\tilde{p}}$ is a diffeomorphism for each $\tilde{p} \in \widetilde{M}$.

Now choose $\tilde{p}_i \in \tilde{\gamma}_i$, and let $\tilde{\gamma}_3$ be minimizing geodesic connecting $\tilde{p}_1$ and $\tilde{p}_2$. Consider the "geodesic quadrilateral" with vertices $\tilde{p}_1, \tilde{p}_2, f(\tilde{p}_1), f(\tilde{p}_2)$. Since f is an isometry, one see that the angle at $\tilde{p}_1$ and the angle at $f(\tilde{p}_1)$ add up to π. Similarly the angle at $\tilde{p}_2$ and the angle at $f(\tilde{p}_2)$ add up to π. On the other hand, one can split the "geodesic quadrilateral" to two "geodesic triangles" by connecting $\tilde{p}_2$ and $f(\tilde{p}_))$. As a consequence, the inner angles of the two "geodesic triangles" add up to at least 2π. [Why "at least"? Think about this!] This contradicts with the lemma above. □

Proof of Preissman's theorem.

As above we denote by $\widetilde{M}$ the universal covering of M, and f_α the deck transformation described above associated to $\alpha \in \pi_1(M)$.

First fix $\alpha \in H$ and let $\tilde{\gamma}$ be the geodesic that is invariant under f_α. Then for any $\beta \in H$, one has $f_{\beta\alpha} = f_{\alpha\beta}$ since H is abelian. So

$$f_\beta(\tilde{\gamma}) = f_\beta(f_\alpha(\tilde{\gamma})) = f_\alpha(f_\beta(\tilde{\gamma})).$$

By the corollary above, one must have

$$f_\beta(\tilde{\gamma}) = \tilde{\gamma}, \quad \forall \beta \in H.$$

As a consequence, γ is invariant under all f_α's for $\alpha \in H$.

Now we denote $\tilde{p}_0 = \tilde{\gamma}(0)$. Since $\tilde{\gamma}$ is invariant under f_β, for each $\beta \in H$, there is a unique $t_\beta \in \mathbb{R}$ so that

$$\tilde{\gamma}(t_\beta) = f_\beta(\tilde{p}_0).$$

Note that this implies

$$\tilde{\gamma}(t_\beta + t) = f_\beta(\tilde{\gamma}(t))$$

for any t, since as t varies, both sides are geodesics with the same initial condition.

Now we define a map $\varphi : H \to \mathbb{R}$ by

$$\varphi(\beta) = t_\beta$$

Claim 1: φ is a group homomorphism:

For any $\beta_1, \beta_2 \in H$,

$$\tilde{\gamma}(t_{\beta_1} + t_{\beta_2}) = f_{\beta_1} \circ f_{\beta_2}(\tilde{p}_0) = f_{\beta_1\beta_2}(\tilde{p}_0) = \tilde{\gamma}(t_{\beta_1\beta_2}).$$

So we have $\varphi(\beta_1\beta_2) = t_{\beta_1\beta_2} = t_{\beta_1} + t_{\beta_2}$.

Claim 2: φ is injective:

Suppose $\varphi(\beta_1) = \varphi(\beta_2)$, i.e. $t_{\beta_1} = t_{\beta_2}$. Then by definition

$$f_{\beta_1}(\tilde{p}_0) = f_{\beta_2}(\tilde{p}_0).$$

So $\tilde{p}_0$ is a fixed point of $f_{\beta_1}^{-1} \circ f_{\beta_2} = f_{\beta_1^{-1}\beta_2}$. This can happen only if $\beta_1^{-1}\beta_2 = e$, i.e. $\beta_1 = \beta_2$.

Claim 3: The image of φ is not dense in $\mathbb{R}$.

Pick a neighborhood U of $p = \pi(\tilde{p}_0)$ so that $\pi^{-1}(U) = \cup_\delta U_\delta$, where each U_δ is diffeomophic to U under π and they are disjoint. By shrinking U one may assume that U is a normal ball of radious r around p. Denote U_0 be the one so that $\tilde{p}_0 \in U_0$. Then for each $\beta \neq e$, $f_\beta(\tilde{p}_0) \notin U_0$. So we see

$$|t_\beta| = d(\tilde{p}_0, f_\beta(\tilde{p}_0)) \geq r$$

for any $\beta \neq e$.

As a consequence of the first two claims, H is an additive subgroup of $\mathbb{R}$. But we know that any additive subgroup of $\mathbb{R}$ is either dense or infinite cyclic. So the theorem is proved. $\square$