LECTURE 20: RAUCH COMPARISON THEOREM AND ITS APPLICATIONS

1. Rauch Comparison Theorem

Let (M, g) and $(\widetilde{M}, \tilde{g})$ be Riemannian manifolds of dimension m. Let

$$\gamma : [0, a] \to M \quad \text{and} \quad \tilde{\gamma} : [0, a] \to \widetilde{M}$$

be normal geodesics with

$$\gamma(0) = p, \tilde{\gamma}(0) = \tilde{p}.$$

For each $t \in [0, a]$, we denote

$$K^-(t) = \min\{K(\Pi_{\gamma(t)}) \mid \dot{\gamma}(t) \in \Pi_{\gamma(t)}\},$$
$$\widetilde{K}^+(t) = \max\{\widetilde{K}(\widetilde{\Pi}_{\tilde{\gamma}(t)}) \mid \dot{\tilde{\gamma}}(t) \in \widetilde{\Pi}_{\tilde{\gamma}(t)}\}.$$

Theorem 1.1 (Rauch comparison theorem)**.** *Let $X, \widetilde{X}$ be Jacobi fields along $\gamma, \tilde{\gamma}$ respectively, such that*

$$①X(0) = 0, \widetilde{X}(0) = 0, \quad ②|\nabla_{\dot{\gamma}(0)}X| = |\widetilde{\nabla}_{\dot{\tilde{\gamma}}(0)}\widetilde{X}|, \quad ③\langle\dot{\gamma}(0), \nabla_{\dot{\gamma}(0)}X\rangle = \langle\dot{\tilde{\gamma}}(0), \widetilde{\nabla}_{\dot{\tilde{\gamma}}(0)}\widetilde{X}\rangle.$$

Assume further that

ⓘ *γ has no conjugate points on $[0, a]$,* ⓘⓘ *$\widetilde{K}^+(t) \le K^-(t)$ holds for all $t \in [0, a]$.*

Then $\tilde{\gamma}$ has no conjugate points on $[0, a]$, and for all $t \in [0, a]$,

$$|X(t)| \le |\widetilde{X}(t)|.$$

We first prove

Lemma 1.2. *Let $X, \widetilde{X}$ be* normal *Jacobi fields along $\gamma, \tilde{\gamma}$ respectively, such that*

$$ⓐX(0) = 0, \widetilde{X}(0) = 0, \quad ⓑ|X(a)| = |\widetilde{X}(a)|.$$

Assume further that ⓘ *and* ⓘⓘ *above holds. Then*

$$I(X, X) \le I(\widetilde{X}, \widetilde{X}).$$

Proof. Let $\{e_1(t), \cdots, e_m(t)\}$ and $\{\tilde{e}_1(t), \cdots, \tilde{e}_m(t)\}$ be orthonormal frames that are parallel along γ and $\tilde{\gamma}$ respectively, such that

$$e_1(t) = \dot{\gamma}(t), \quad \tilde{e}_1(t) = \dot{\tilde{\gamma}}(t)$$

and

$$e_2(a) = X(a)/\alpha, \quad \tilde{e}_2(a) = \widetilde{X}(a)/\alpha,$$

where $\alpha = |X(a)| = |\widetilde{X}(a)| \neq 0$ since γ has no conjugate point. If we denote

$$X(t) = X^i(t)e_i(t), \quad \widetilde{X}(t) = \widetilde{X}^i(t)\tilde{e}_i(t)$$

respectively, then obviously we have

- $X^i(0) = \widetilde{X}^i(0) = 0$ for all i,
- $X^2(a) = \widetilde{X}^2(a) = \alpha$ and $X^i(a) = \widetilde{X}^i(a) = 0$ for all $i \neq 2$,
- $X^1(t) = \widetilde{X}^1(t) = 0$ for all $t \in [0, a]$.

We define a vector field Y along γ by

$$Y = \widetilde{X}^i(t)e_i(t).$$

Then $Y(0) = 0$, $Y(a) = X(a)$. So by corollary 2.4 in lecture 17,

$$I(X, X) \leq I(Y, Y).$$

On the other handside,

$$\begin{aligned}
I(Y, Y) &= \int_0^a \left(|\nabla_{\dot\gamma} Y|^2 - Rm(\dot\gamma, Y, \dot\gamma, Y)\right) dt \\
&= \int_0^a \left(\sum (\dot{\tilde{X}}^i(t))^2 - \sum (\tilde{X}^i(t))^2 K(\dot\gamma, Y)\right) dt \\
&\leq \int_0^a \left(\sum (\dot{\tilde{X}}^i(t))^2 - \sum (\tilde{X}^i(t))^2 K^-(t)\right) dt \\
&\leq \int_0^a \left(\sum (\dot{\tilde{X}}^i(t))^2 - \sum (\tilde{X}^i(t))^2 \widetilde{K}^+(t)\right) dt \\
&\leq \int_0^a \left(|\widetilde\nabla_{\dot{\tilde\gamma}} \widetilde{X}|^2 - \widetilde{Rm}(\dot{\tilde\gamma}, \widetilde{X}, \dot{\tilde\gamma}, \widetilde{X})\right) dt \\
&= I(\widetilde{X}, \widetilde{X}).
\end{aligned}$$

It follows that $I(X, X) \leq I(\widetilde{X}, \widetilde{X})$. □

Proof of Rauch Comparison Theorem. The conditions ① and ③ implies that the tangential components of X and $\widetilde{X}$ have the same length at correponding points. So WLOG, we may assume that both X and $\widetilde{X}$ are *normal* Jacobi fields. We denote

$$u(t) = |X(t)|^2, \quad \tilde{u}(t) = |\widetilde{X}(t)|^2.$$

Then $\tilde{u}(t)/u(t)$ is well-defined, and by L'Hospital's rule,

$$\lim_{t\to 0} \frac{\tilde{u}(t)}{u(t)} = \lim_{t\to 0} \frac{\ddot{\tilde{u}}(t)}{\ddot{u}(t)} = \frac{|\nabla_{\dot\gamma(0)} X|^2}{|\nabla_{\dot{\tilde\gamma}(0)} \widetilde{X}|^2} = 1.$$

Therefore, to prove $|X| \leq |\widetilde{X}|$, it is enough to prove $\frac{d}{dt}\frac{\tilde{u}(t)}{u(t)} \geq 0$, or equivalently,

$$\dot{\tilde{u}}(t)u(t) - \tilde{u}(t)\dot{u}(t) \geq 0.$$

Since γ has no conjugate point, $u(t) > 0$ for all $t \in (0, a]$. Let $c \leq a$ be the greatest number so that $\tilde{u}(t) > 0$ on $(0, c)$. For any $b \in (0, c)$, we define

$$X_b(t) = \frac{X(t)}{|X(b)|}, \quad \widetilde{X}_b(t) = \frac{\widetilde{X}(t)}{|\widetilde{X}(b)|}.$$

Applying the previous lemma to $X_b, \widetilde{X}_b$ on $[0, b]$, we get

$$I(X_b, X_b) \leq I(\widetilde{X}_b, \widetilde{X}_b).$$

Recall that after integration by parts

$$I(X, Y) = -\int_0^a \langle \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X + R(\dot{\gamma}, X)\dot{\gamma}, Y\rangle dt + \langle \nabla_{\dot{\gamma}} X, Y\rangle|_0^a.$$

So

$$I(X_b, X_b) = \langle \nabla_{\dot{\gamma}(b)} X_b, X_b(b)\rangle \quad \text{and} \quad I(\widetilde{X}_b, \widetilde{X}_b) = \langle \widetilde{\nabla}_{\dot{\tilde{\gamma}}(b)} \widetilde{X}_b, \widetilde{X}_b(b)\rangle.$$

It follows

$$\langle \nabla_{\dot{\gamma}(b)} X_b, X_b(b)\rangle \leq \langle \widetilde{\nabla}_{\dot{\tilde{\gamma}}(b)} \widetilde{X}_b, \widetilde{X}_b(b)\rangle,$$

i.e.

$$\frac{1}{2}\frac{\dot{u}(b)}{u(b)} = \frac{\langle \nabla_{\dot{\gamma}(b)} X, X(b)\rangle}{\langle X(b), X(b)\rangle} \leq \frac{\langle \widetilde{\nabla}_{\dot{\tilde{\gamma}}(b)} \widetilde{X}, \widetilde{X}(b)\rangle}{\langle \widetilde{X}(b), \widetilde{X}(b)\rangle} = \frac{1}{2}\frac{\dot{\tilde{u}}(b)}{\tilde{u}(b)}.$$

So for any $t \in (0, c)$, we have $\frac{\dot{u}(t)}{u(t)} \leq \frac{\dot{\tilde{u}}(t)}{\tilde{u}(t)}$. This is exactly what we need.

To summary, we proved that $|X(t)| \leq |\widetilde{X}(t)|$ for $t \in (0, c)$. If $c < a$, then $|\widetilde{X}(c)| \geq |X(c)| > 0$, contradicting with the choice of c. So we must have $c = a$. In particular, $\tilde{\gamma}$ has no conjugate points on $[0, a]$. This completes the proof. □

Recall that any Jacobi field is a variation field of some geodesic variation. In particular, if X is a Jacobi field along $\gamma : [0, a] \to M$ with

$$X(0) = 0, \quad \nabla_{\dot{\gamma}(0)} X = X_p,$$

then the corresponding variation can be chosen as

$$\gamma_s(t) = \exp_p(t(\dot{\gamma}(0) + sX_p)),$$

and as a consequence, the Jacobi field X is explicitly given by

$$X(t) = t(d\exp_p)_{t\dot{\gamma}(0)} X_p.$$

One can rewrite Rauch comparison theorem above as

Theorem 1.3 (Rauch comparison theorem, Second form). *Suppose* ⓘ *and* ⓘⓘ *holds. Denote* $p = \gamma(0)$ *and* $\tilde{p} = \tilde{\gamma}(0)$, *and suppose* $X_p \in T_pM, \widetilde{X}_{\tilde{p}} \in T_{\tilde{p}}\widetilde{M}$ *satisfies*

$$\langle X_p, \dot{\gamma}(0)\rangle = \langle \widetilde{X}_{\tilde{p}}, \dot{\tilde{\gamma}}(0)\rangle, \quad |X_p| = |\widetilde{X}_{\tilde{p}}|.$$

Then $|(d\exp_p)_{t\dot{\gamma}(0)} X_p| \leq |(d\exp_{\tilde{p}})_{t\dot{\tilde{\gamma}}(0)} \widetilde{X}_p|$

Remark. If we replace the inequality $K^- \geq \widetilde{K}^+$ in ⓘⓘ by the strict one

$$K^- > \widetilde{K}^+,$$

then the inequality in lemma 1.2 is strict. As a consequence, the inequalities in theorems 1.1 and 1.3 are strict for $t > 0$.

2. Some Direct Applications

In applications, we will mainly compare a given Riemannian manifold with with another "model" Riemannian manifold of constant curvature. Here we list several interesting applications, and leave the proofs as exercises.

Corollary 2.1. *Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature. Then for any $p \in M$, any $X_p \in T_pM$ and $Y_p \in T_pM = T_{X_p}(T_pM)$,*

$$|(d\exp_p)_{X_p}(Y_p)| \geq |Y_p|.$$

In particular, for any curve γ in T_pM, one has

$$L(\gamma) \leq L(\exp_p \circ \gamma).$$

Proof. Apply theorem 1.3 to (M, g) and (T_pM, g_p). □

Corollary 2.2. *Let (M, g) be a complete simply connected Riemannian manifold with non-positive sectional curvature, and consider a geodesic triangle in M whose side lengths are a, b, c with opposite angles A, B, C respectively. Then*

(1) $a^2 + b^2 - 2ab\cos C \leq c$,
(2) $A + B + C \leq \pi$.

Moreover, if M has negative sectional curvature, then the inequalities are strict.

Proof. (1) Denote the vertex at the angle C by p. In the tangent space T_pM, draw a triangle $\triangle OPQ$, where O is the origin of T_pM, so that

$$|OP| = a, \quad |OQ| = b \quad \text{and} \quad \angle O = C.$$

Let η be the pre-image of the geodesic c in T_pM. Then

$$|PQ| \leq L(\eta) \leq c,$$

where the second inequality follows from corollary 2.1. Now apply the Euclidean cosine law to $\triangle OPQ$.

(2) Construct a triangle in $\mathbb{R}^2$ whose side lengths are a, b, c. This is possible since a, b, c are lengths of sides of a geodesic triangle (meaning each side is a shortest geodesic connecting the corresponding vertices). Denote the corresponding angles of this new triangle by A', B', C'. Then by (1), we have $C \leq C'$. Similarly $A \leq A'$ and $B \leq B'$.

If $K < 0$, then according to the remark after theroem 1.3, the inequality in lemma 2.1 is also strict for $X_p, Y_p \neq 0$. So the conclusion follows. □

Corollary 2.3. *Suppose the sectional curvature of* (M, g) *satisfies*

$$0 < C_1 \le K \le C_2,$$

where C_1, C_2 *are constants. Let* γ *be any geodesic in* M*. Then the distance* D *between any two consecutive conjugate points of* γ *satisfies*

$$\frac{\pi}{\sqrt{C_2}} \le D \le \frac{\pi}{\sqrt{C_1}}.$$

Proof. This follows from the explicit formula for Jacobi fields on spheres. □

Remark. One should compare this with the Sturm comparison theorem in ODE.