

## LECTURE 21: THE HESSIAN, LAPLACE AND TOPOROGOV COMPARISON THEOREMS

### 1. THE HESSIAN COMPARISON THEOREM

We recall from last lecture that

$$\begin{aligned} K^-(t) &= \min\{K(\Pi_{\gamma(t)}) \mid \dot{\gamma}(t) \in \Pi_{\gamma(t)}\}, \\ \tilde{K}^+(t) &= \max\{K(\tilde{\Pi}_{\tilde{\gamma}(t)}) \mid \dot{\tilde{\gamma}}(t) \in \tilde{\Pi}_{\tilde{\gamma}(t)}\}. \end{aligned}$$

and we have shown (lemma 1.2 in lecture 20) that if  $X$  and  $\tilde{X}$  are *normal* Jacobi fields along normal geodesics  $\gamma : [0, a] \rightarrow M$  and  $\tilde{\gamma} : [0, a] \rightarrow \tilde{M}$  respectively, such that

$$\textcircled{a} X(0) = 0, \tilde{X}(0) = 0, \quad \textcircled{b} |X(a)| = |\tilde{X}(a)|.$$

and assume further that

$$\textcircled{i} \gamma \text{ has no conjugate points on } [0, a], \quad \textcircled{ii} \tilde{K}^+(t) \leq K^-(t) \text{ holds for all } t \in [0, a].$$

then

$$I(X, X) \leq I(\tilde{X}, \tilde{X}).$$

Using this we showed that the Jacobi fields with “the same initial conditions” are comparable.

Today we would like to compare the distance functions on different manifolds. We let  $(M, g)$  be a complete Riemannian manifold. We fix  $p \in M$  and consider the distance function

$$d_p : M \rightarrow \mathbb{R}, \quad d_p(q) = \text{dist}(p, q).$$

Then we have (c.f. lecture 16 and PSet 3)

- $d_p$  is a Lipschitz continuous function.
- $d_p$  is smooth on  $M \setminus \text{Cut}(p) \cup \{p\}$ .
- For each  $q \in M \setminus \text{Cut}(p) \cup \{p\}$ , the gradient vector of  $d_p$  at  $q$  is

$$(\nabla d_p)(q) = \dot{\gamma}^q(d_p(q)),$$

where  $\gamma^q$  is the unique normal minimizing geodesic from  $p$  to  $q$ .

- For any  $q \in M \setminus \text{Cut}(p) \cup \{p\}$  and  $Y_q \in T_q M$ , the Hessian of  $d_p$  is given by

$$\nabla^2 d_p(Y_q, Y_q) = d_p(q) \langle \nabla_{\dot{\gamma}^q(d_p(q))} X, X(d_p(q)) \rangle.$$

where  $X$  is the Jacobi field along  $\gamma^q$  such that  $X(0) = 0, X(d_p(q)) = Y_q$ .

Since  $\nabla d_p$  has length 1 almost everywhere, to compare the distance functions one must compare their “second order derivatives”, i.e. compare the Hessians. We recall from lecture 5 that the Hessian  $\nabla^2 f$  of a smooth function  $f$  on  $M$  is

$$\nabla^2 f(X, Y) = \langle \nabla_X \nabla f, Y \rangle.$$

We have shown that it is a symmetric  $(0, 2)$  tensor, and

$$\Delta f = \text{Tr}(\nabla^2 f).$$

We are ready to state

**Theorem 1.1** (The Hessian Comparison Theorem). *Let  $(M, g), (\widetilde{M}, \widetilde{g})$  be complete Riemannian manifolds,  $\gamma : [0, b] \rightarrow M$  and  $\widetilde{\gamma} : [0, b] \rightarrow \widetilde{M}$  be minimizing normal geodesics in  $M$  and  $\widetilde{M}$  respectively, so that*

$$\widetilde{K}^+(t) \leq K^-(t) \text{ holds for all } t \in [0, b].$$

*Denote  $q = \gamma(a)$  and  $\widetilde{q} = \widetilde{\gamma}(a)$  ( $a < b$ ). Suppose  $X_q \in T_q M$  and  $\widetilde{X}_{\widetilde{q}} \in T_{\widetilde{q}} \widetilde{M}$  satisfy*

$$\langle X_q, \dot{\gamma}(a) \rangle = \langle \widetilde{X}_{\widetilde{q}}, \dot{\widetilde{\gamma}}(a) \rangle, \quad |X_q| = |\widetilde{X}_{\widetilde{q}}|.$$

*Then*

$$\nabla^2 d_p(X_q, X_q) \leq \widetilde{\nabla}^2 \widetilde{d}_{\widetilde{p}}(\widetilde{X}_{\widetilde{q}}, \widetilde{X}_{\widetilde{q}}).$$

*Proof.* For any  $Y_q \in T_q M$ , we have

$$\nabla^2 d_p(\dot{\gamma}(a), Y_q) = \langle \nabla_{\dot{\gamma}(a)} \nabla d_p, Y_q \rangle = \langle \nabla_{\dot{\gamma}(a)} \dot{\gamma}, Y_q \rangle = 0.$$

The same relation also works for  $\widetilde{M}$ . So WLOG, we can assume

$$X_q \perp \dot{\gamma}(a) \quad \text{and} \quad \widetilde{X}_{\widetilde{q}} \perp \dot{\widetilde{\gamma}}(a).$$

Let  $X, \widetilde{X}$  be normal Jacobi fields along  $\gamma, \widetilde{\gamma}$  respectively, so that

$$X(0) = 0, X(a) = X_q \quad \text{and} \quad \widetilde{X}(0) = 0, \widetilde{X}(a) = \widetilde{X}_{\widetilde{q}}.$$

Since  $|X(a)| = |\widetilde{X}(a)|$ , we get (use the lemma 1.2 in lecture 20)

$$I(X, X) \leq I(\widetilde{X}, \widetilde{X}).$$

Now the theorem follows from

$$\nabla^2 d_p(X_q, X_q) = a \langle \nabla_{\dot{\gamma}(a)} X, X(a) \rangle = a I(X, X).$$

□

Since  $\Delta = \text{Tr} \nabla^2$ , by taking trace we immediately get

**Corollary 1.2.** *Under the same assumptions as in theorem 1.1, we have*

$$\Delta d_p(q) \leq \widetilde{\Delta} \widetilde{d}_{\widetilde{p}}(\widetilde{q}).$$

## 2. THE LAPLACE COMPARISON THEOREM

In many applications, one would like to weaken the conditions on the sectional curvatures in corollary 1.2 to conditions on the Ricci curvatures. This is not always true. However, we will prove below that this is possible if we assume that  $\widetilde{M}$  has constant sectional curvature.

Let's fix some notions. For any  $k \in \mathbb{R}$ , we will denote by  $M_k^m$  the  $m$ -dimensional space form of constant curvature  $k$ . In other words,  $M_k^m = S^m(k)$  or  $\mathbb{R}^m$  or  $\mathbb{H}^m(k)$ , depending on the sign of  $k$ .

**Theorem 2.1** (The Laplace Comparison Theorem). *Let  $(M, g)$  be a Riemannian manifold of dimension  $m$  whose Ricci curvature is bounded below,*

$$\text{Ric}(M, g) \geq (m - 1)k.$$

*Let  $\gamma : [0, b] \rightarrow M$ ,  $\tilde{\gamma} : [0, b] \rightarrow M_k^m$  be minimizing normal geodesics in  $M$  and  $M_k^m$  starting from  $\gamma(0) = p$  and  $\tilde{\gamma}(0) = \tilde{p}$  respectively. Then*

$$\Delta d_p(\gamma(t)) \leq \widetilde{\Delta} \tilde{d}_{\tilde{p}}(\tilde{\gamma}(t)).$$

*Remark.* One can replace  $M_k^m$  by a Riemannian manifold  $\widetilde{M}$  which has constant sectional curvature  $k$  along the geodesic  $\tilde{\gamma}$ , and assume that for any  $t \in [0, b]$ ,

$$\text{Ric}(\dot{\gamma}(t), \dot{\gamma}(t)) \geq (m - 1)k = \widetilde{\text{Ric}}(\dot{\tilde{\gamma}}(t), \dot{\tilde{\gamma}}(t)).$$

The proof is identically the same as below.

*Proof.* Fix  $t$ . Let  $\{e_1, \dots, e_m\}$  be an orthonormal frame that is parallel along  $\gamma$ , such that  $e_1 = \dot{\gamma}$ . For any  $i \geq 2$  let  $X_i(\tau)$  be the normal Jacobi field along  $\gamma|_{[0, t]}$  with  $X_i(0) = 0$  and  $X_i(t) = e_i(t)$ . Then

$$\Delta d_p(\gamma(t)) = \sum_{i=2}^m \nabla^2 d_p(e_i(t), e_i(t)) = t \sum_{i=2}^m I(X_i, X_i)$$

Similarly

$$\widetilde{\Delta} \tilde{d}_{\tilde{p}}(\tilde{\gamma}(t)) = t \sum_{i=2}^m I(\tilde{X}_i, \tilde{X}_i),$$

where  $\tilde{X}_i$  is the normal Jacobi field along  $\tilde{\gamma}|_{[0, t]}$  such that  $\tilde{X}_i(0) = 0$ ,  $\tilde{X}_i(t) = \tilde{e}_i(t)$ . It remains to prove

$$\sum_{i=2}^m I(X_i, X_i) \leq \sum_{i=2}^m I(\tilde{X}_i, \tilde{X}_i).$$

We shall apply the same trick that we played in the proof of lemma 1.2 in lecture 20. In other words, we write for each  $0 \leq \tau \leq t$ ,

$$X_i(\tau) = \sum a_i^j(\tau) e_j(\tau).$$

Then by assumptions  $X_i(0) = 0$  and  $X_i(t) = e_i(t)$ , we see

$$a_i^j(0) = 0 \quad \text{and} \quad a_i^j(t) = \delta_i^j.$$

On the other hand, since  $M_k^m$  has constant sectional curvature  $k$ , our computations in lecture 12 together with the conditions  $\tilde{X}_i(0) = 0, \tilde{X}_i(t) = \tilde{e}_i(t)$  implies

$$\tilde{X}_i(\tau) = \sum \delta_i^j \frac{\text{sn}_k(\tau)}{\text{sn}_k(t)} \tilde{e}_j(\tau) = \frac{\text{sn}_k(\tau)}{\text{sn}_k(t)} \tilde{e}_i(\tau),$$

where  $\text{sn}_k(\tau) = \frac{\sin(\sqrt{k}\tau)}{\sqrt{k}}$  or  $\tau$  or  $\frac{\sinh(\sqrt{-k}\tau)}{\sqrt{-k}}$ .

Now for each  $2 \leq i \leq m$  we define on  $\gamma|_{[0,t]}$  a vector field

$$X_i'(\tau) = \frac{\text{sn}_k(\tau)}{\text{sn}_k(t)} e_i(\tau)$$

Then  $X_i'$  has the same boundary condition as  $X_i$ . Since  $X_i$ 's are Jacobi fields, we have

$$I(X_i, X_i) \leq I(X_i', X_i').$$

Finally we the conclusion follows from

$$\begin{aligned} \sum I(X_i', X_i') &= \sum \int_0^t (|\nabla_{\dot{\gamma}} X_i'|^2 - \text{Rm}(\dot{\gamma}, X_i', \dot{\gamma}, X_i')) d\tau \\ &= \sum_i \int_0^t \left( \left( \frac{\text{sn}_k'(\tau)}{\text{sn}_k(t)} \right)^2 - \left( \frac{\text{sn}_k(\tau)}{\text{sn}_k(t)} \right)^2 K(\dot{\gamma}, e_i) \right) d\tau \\ &= \int_0^t \left( (m-1) \left( \frac{\text{sn}_k'(\tau)}{\text{sn}_k(t)} \right)^2 - \left( \frac{\text{sn}_k(\tau)}{\text{sn}_k(t)} \right)^2 \text{Ric}(\dot{\gamma}) \right) dt \\ &\leq \int_0^t \left( (m-1) \left( \frac{\text{sn}_k'(\tau)}{\text{sn}_k(t)} \right)^2 - \left( \frac{\text{sn}_k(\tau)}{\text{sn}_k(t)} \right)^2 (m-1)k \right) dt \\ &= \int_0^t \left( (m-1) \left( \frac{\text{sn}_k'(\tau)}{\text{sn}_k(t)} \right)^2 - \left( \frac{\text{sn}_k(\tau)}{\text{sn}_k(t)} \right)^2 \widetilde{\text{Ric}}(\dot{\gamma}) \right) dt \\ &= \sum \int_0^t (|\nabla_{\dot{\gamma}} \tilde{X}_i|^2 - \widetilde{\text{Rm}}(\dot{\gamma}, \tilde{X}_i, \dot{\gamma}, \tilde{X}_i)) d\tau \\ &= \sum I(\tilde{X}_i, \tilde{X}_i). \end{aligned}$$

□

### 3. THE TOPOROGOV COMPARISON THEOREM

Let  $(M, g)$  be complete. Recall that a *geodesic triangle*  $\triangle ABC$  consists of three points  $A, B, C$  in  $M$  (which are called the *vertices*) and three minimizing geodesics (which are called the *sides*)  $\gamma_{AB}, \gamma_{BC}, \gamma_{CA}$  joining each two of them. If only two

sides, say  $\gamma_{AB}$  and  $\gamma_{AC}$ , are minimal, while the third side is a geodesic, which may be not minimal, but still satisfies the triangle inequality

$$L(\gamma_{BC}) \leq L(\gamma_{AB}) + L(\gamma_{AC}),$$

then we will call  $\triangle ABC$  a *generalized geodesic triangle*.

Similarly we can define a *geodesic hinge*  $\angle BAC$ , which consists of a point  $A$  in  $M$  (which is again called the *vertex*) and two minimal geodesics  $\gamma_{AB}, \gamma_{AC}$  (called the *sides*) emanating from  $A$ , with end points  $B$  and  $C$  in  $M$ . If one side, say  $\gamma_{AB}$ , is minimal, while the other side  $\gamma_{AC}$  is not, we will call  $\angle BAC$  a *generalized geodesic hinge*.

In what follows when we say hinge or triangle, we always mean generalized geodesic hinge or generalized geodesic triangle.

**Lemma 3.1.** *Let  $(M, g)$  be a complete Riemannian manifold of dimension  $m$  whose sectional curvature  $K \geq k$ . Then*

- (1) *For each generalized geodesic hinge  $\angle BAC$  in  $M$ , there is a hinge  $\angle \tilde{B}\tilde{A}\tilde{C}$  in  $M_k^m$  with same angle and the corresponding sides are of same length as  $\angle BAC$ .*
- (2) *For each generalized geodesic triangle  $\triangle ABC$  in  $M$ , there is a triangle  $\triangle \tilde{A}\tilde{B}\tilde{C}$  in  $M_k^m$  whose corresponding sides have the same length as  $\triangle ABC$ .*

*Proof.* For  $k = 0$  and  $k < 0$ , there is a unique geodesic between any two points, and the lemma is clear. So in what follows we assume  $k > 0$ . According to the theorem of Bonnet-Myers, one has  $\text{diam}M \leq \frac{\pi}{\sqrt{k}}$ . More over, if  $\text{diam}M = \frac{\pi}{\sqrt{k}}$ , then according to the maximal diameter theorem of Cheng that we will later,  $(M, g)$  is isometric to the standard sphere of radius  $\frac{1}{\sqrt{k}}$  and thus the conclusion holds. So WLOG, we may assume that any minimizing geodesic has length less than  $\frac{\pi}{\sqrt{k}}$ .

In the hinge case, we can fix any  $\tilde{A} \in M_k^m$ , and choose any  $\tilde{B}$  so that

$$\text{dist}(\tilde{A}, \tilde{B}) = L(\gamma_{AB}) < \frac{\pi}{\sqrt{k}},$$

and join them by the unique minimizing geodesic. Then we choose a direction at  $\tilde{A}$  so that the angle with the geodesic  $\tilde{\gamma}_{\tilde{A}\tilde{B}}$  is  $\angle A$ . Use this direction one can generate a normal geodesic emanating from  $\tilde{A}$ . Take the point  $\tilde{C}$  to be the point on this geodesic with parameter  $L(\gamma_{AC})$ . (Note that  $\tilde{\gamma}_{\tilde{A}\tilde{C}}$  is minimizing if  $\gamma_{AC}$  is minimizing.)

In the triangle case, we pick  $\tilde{B}$  and  $\tilde{C}$  as above, and then consider the two distance spheres  $\partial B_{\tilde{B}}(\text{dist}(A, B))$  and  $\partial B_{\tilde{C}}(\text{dist}(A, C))$ . One can easily check that these two spheres are non-empty and must intersect. We then take any point  $\tilde{A}$  from the intersection and connect it to  $\tilde{B}$  and  $\tilde{C}$  by minimizing geodesics.  $\square$

One can think of the following Toporogov comparison theorem as a global version of Hessian comparison theorem, where we can actually compare the distance function instead of only comparing their Hessian.

**Theorem 3.2** (Toporogov Comparison Theorem). *Let  $(M, g)$  be a complete Riemannian manifold with sectional curvature  $K \geq k$ . Then*

- (1) (**Hinge Version**) *Let  $\angle BAC$  be a hinge in  $M$  and  $\angle \tilde{B}\tilde{A}\tilde{C}$  a comparing hinge in  $M_k^m$ . Then  $\text{dist}(B, C) \leq \text{dist}(\tilde{B}, \tilde{C})$ .*
- (2) (**Triangle Version**) *Let  $\triangle ABC$  be a triangle in  $M$  and  $\triangle \tilde{A}\tilde{B}\tilde{C}$  a comparing triangle in  $M_k^m$ . Then the three angles in  $\triangle ABC$  are greater than the corresponding angles in  $\triangle \tilde{A}\tilde{B}\tilde{C}$ .*

The proof will be left as a possible final topic.

*Remarks.* (1) There is no analogous theorem for the case  $K \leq k$ .

(2) One can prove that the hinge version and the triangle version of Toporogov comparison theorem are equivalent.

(3) One can replace  $M_k^m$  by  $M_k^2$ .

(4) In the case  $k > 0$ , one can show that the perimeter of the generalized geodesic triangle  $\triangle ABC$  is no more than  $\frac{2\pi}{\sqrt{k}}$ .

As an immediate application, we will prove

**Theorem 3.3** (Gromov). *There is a constant  $C = C(m)$  so that for any complete Riemannian manifold  $(M, g)$  with  $K \geq 0$ , the fundamental group  $\pi_1(M)$  is generated by no more than  $C(m)$  generators.*

*Proof.* We will consider  $\pi_1(M)$  as the group of Deck transformations on the universal covering  $\tilde{M}$ . Fix  $\tilde{p} \in \tilde{M}$  and choose inductively a generating set of  $\pi_1(M)$  as follows:

- We first choose  $e \neq g_1 \in \pi_1(M)$  so that for all  $g \in \pi_1(M) \setminus \{e\}$ ,

$$d(\tilde{p}, g_1 \cdot \tilde{p}) \leq d(\tilde{p}, g \cdot \tilde{p}).$$

- Suppose  $g_1, \dots, g_{k-1}$  are chosen. We then choose  $g_k \notin \langle g_1, \dots, g_{k-1} \rangle$  so that

$$d(\tilde{p}, g_k \cdot \tilde{p}) \leq d(\tilde{p}, g \cdot \tilde{p})$$

holds for all  $g \in \pi_1(M) \setminus \langle g_1, \dots, g_{k-1} \rangle$ .

Now let  $\tilde{\gamma}_k$  be a minimal geodesic in  $(\tilde{M}, \tilde{g})$  from  $\tilde{p}$  to  $g_k \cdot \tilde{p}$ . We claim that the angle between any two such minimal geodesics is at least  $\frac{\pi}{3}$ . Then the conclusion follows.

We prove the claim by contradiction. Suppose the angle between  $\tilde{\gamma}_k$  and  $\tilde{\gamma}_{k+l}$  is less than  $\frac{\pi}{3}$ . For simplicity we denote  $\tilde{l}_k = d(\tilde{p}, g_k \cdot \tilde{p})$ . Then according to the Toporogov comparison theorem,

$$d(g_{k+l} \cdot \tilde{p}, g_k \cdot \tilde{p})^2 < l_k^2 + l_{k+l}^2 - l_k l_{k+l} \leq l_{k+l}^2.$$

This implies

$$d(\tilde{p}, g_{k+l}^{-1} g_k \cdot \tilde{p}) = d(g_{k+l} \cdot \tilde{p}, g_k \cdot \tilde{p}) < l_{k+l} = d(\tilde{p}, g_{k+l} \cdot \tilde{p}),$$

which contradicts with the choice of  $g_{k+l}$ .  $\square$

*Remark.*

- By the same way one can prove the following theorem of Gromov:

**Theorem 3.4** (Gromov). *For  $k$  negative, there is a constant  $C = C(m, k, D)$  so that for any complete Riemannian manifold  $(M, g)$  with  $K \geq k$  and  $\text{diam}(M, g) \leq D$ , the fundamental group  $\pi_1(M)$  is generated by no more than  $C(m, k, D)$  generators.*

Note that a bound on diameter is needed. To see this, one can look at the example of surface of genus  $g$ .

- One of the “most beautiful theorem” in Riemannian geometry is

**Theorem 3.5** (Gromov). *There is a constant  $C = C(m)$  so that for any complete Riemannian manifold  $(M, g)$  with  $K \geq 0$ , the total betti number  $b(M) = \sum b_i(M) \leq C(m)$ . [Similarly one can state a version with  $K \geq k$ , but with the additional condition  $\text{diam}(M, g) \leq D$ . ]*

Gromov conjectured  $C(m) = 2^m$ . [This is the total Betti number for  $T^m$ .]