LECTURE 22: THE SPHERE THEOREM

1. The estimates of the injectivity radius

Recall that the injectivity radius of a Riemannian manifold (M, g) is

$$\mathrm{inj}(M, g) = \inf_{p \in M} \mathrm{inj}_p(M, g),$$

where $\mathrm{inj}_p(M, g)$ is the injectivity radius at p, defined by

$$\mathrm{inj}_p(M, g) = \sup\{r \mid \exp_p \text{ is diffeomorphism on } B_r(0) \subset T_pM\}.$$

Also recall from PSet 3 that $\exp_p$ is a diffeomorphism onto $M \setminus \mathrm{Cut}(p)$, and $\mathrm{Cut}(p)$ is closed in M. So

$$\mathrm{inj}_p(M, g) = \mathrm{d}(p, \mathrm{Cut}(p)) \qquad \text{and} \qquad \mathrm{inj}(M, g) = \inf_{p \in M} \mathrm{d}(p, \mathrm{Cut}(p)).$$

This fact has the following quite interesting application:

Proposition 1.1. *Let $p \in M$, and $q \in \mathrm{Cut}(p)$ so that $\mathrm{d}(p, q) = \mathrm{d}(p, \mathrm{Cut}(p))$. Then one of the following assertions hold:*

(1) *q is conjugate to p along a minimizing geodesic γ joining p to q,*
(2) *there exists exactly two normal minimizing geodesics γ, σ joining p to q.*

Moreover, in the second case, we must have $\dot\gamma(l) = -\dot\sigma(l)$, where $l = \mathrm{d}(p, q)$.

Proof. Suppose q is not conjugate to p along any minimizing geodesic. Then there exist two minimizing normal geodesic γ and σ of length $l = d(p, q)$ joining p to q.

Assume $\dot\gamma(l) \neq -\dot\sigma(l)$. Then there exists a unit vector $X_q \in S_qM$ so that

$$\langle X_q, \dot\gamma(l)\rangle < 0, \quad \langle X_q, \dot\sigma(l)\rangle < 0.$$

Since q is not conjugate to p along γ, there exists a neighborhood U of $l\dot\gamma(0)$ in T_pM so that $\exp_p|_U$ is a diffeomorphism. For s small, let

$$\xi(s) = (\exp_p|_U)^{-1} \exp_q(sX_q).$$

Using ξ one can construct a geodesic variation of γ by

$$\gamma_s(t) = \exp_p\left(\frac{t}{l}\xi(s)\right).$$

Then by the first variation formula,

$$\frac{dL(\gamma_s)}{ds}(0) = \langle X_q, \dot\gamma(l)\rangle < 0.$$

So for s small enough, $L(\gamma_s) < L(\gamma)$. By the same argument, we can construct a geodesic variation

$$\sigma_s(t) = \exp_p\left(\frac{t}{l}(\exp_p|_V)^{-1}\exp_q(sX_q)\right)$$

of σ so that $L(\sigma_s) < L(\sigma)$ for s small, where V is a neighborhood of $l\dot{\sigma}(0)$ in T_qM so that $\exp_p$ is a diffeomorphism on it. Note that for each s, both γ_s and σ_s are geodesics from p to $\exp_q(sX_q)$. However, for s small,

$$l_s := \mathrm{d}(p, \exp_q(sX_q)) \leq L(\gamma_s) < \mathrm{dist}(p,q) = \mathrm{inj}_p(M,g) = l,$$

so $\exp_p$ is not injective on $B_{\frac{l_s+l}{2}}(0) \subset T_pM$. A contradiction. □

As a corollary, we have

Corollary 1.2 (Klingenberg)**.** *Let (M,g) be a compact Riemannian manifold whose sectional curvature satisfies $K \leq C$ for some constant C. Then either*

$$\mathrm{inj}(M,g) \geq \frac{\pi}{\sqrt{C}}$$

or there exists a closed geodesic γ in M whose length is minimum among all closed geodesics, such that

$$\mathrm{inj}(M,g) = \frac{1}{2}L(\gamma).$$

Proof. Take $p \in M$ and $q \in \mathrm{Cut}(p)$ so that $d(p,q) = \mathrm{inj}(M,g)$. If q is conjugate to p along some minimizing geodesic, then by corollary 2.3 in lecture 20,

$$\mathrm{inj}(M,g) = \mathrm{dist}(p,q) \geq \frac{\pi}{\sqrt{C}}.$$

If q is not conjugate to p, then there exists two normal minimizing geodesics σ, τ joining p to q so that $\dot{\sigma}(l) = -\dot{\tau}(l)$, where $l = \mathrm{dist}(p,q)$. Since p is also a cut point of q but not a conjugate point of p, and by definition p realize the distance from q to $\mathrm{Cut}(q)$. So we also have $\dot{\sigma}(0) = -\dot{\tau}(0)$. So σ and τ together form a closed geodesic. If we denote this closed geodesic by γ, then

$$\mathrm{inj}(M,g) = \frac{1}{2}L(\gamma).$$

Obviously γ has to have minimal length among all closed geodesics. Otherwise if there is another closed geodesic γ' with length $L(\gamma') < L(\gamma)$, and let p', q' be two "antipodal" points on γ', i.e. $d(p',q') = \frac{1}{2}L(\gamma')$, then by definition there is a point q'' on γ' which lies in $\mathrm{Cut}(p')$, and

$$d(p',q'') \leq \frac{1}{2}L(\gamma').$$

Contradiction. □

One of the crucial step in proving the sphere theorem (and its generalizations) is the following injectivity radius estimate by Klingenberg.

Theorem 1.3 (Klingenberg)**.** *Let (M, g) be a complete simply connected Riemannian manifold such that $\frac{1}{4} < K \leq 1$. Then* $\operatorname{inj}(M, g) \geq \pi$.

We will only prove the theorem for the case $m = \dim M$ is even. The proof of more general case demands Morse theory, (and will be a possible final project).

Proof of Klingenberg's estimate for the case $m = \dim M$ even.

By Bonnet-Myers' theorem, M is compact. So there exists $p \in M$ and $q \in \operatorname{Cut}(p)$ so that $\operatorname{dist}(p, q) = \operatorname{inj}(M, g)$. Suppose the theorem does not hold, i.e. $l = \operatorname{dist}(p, q) < \pi$. Then by corollary 2.3 in lecture 20, q is not conjugate to p. So according to corollary 1.2, there exists a closed normal geodesic γ in M passing $p = \gamma(0)$ and $q = \gamma(l)$ whose length is $L(\gamma) = 2l < 2\pi$.

By repeating the proof of Synge's theorem, we can find a vector field $X(t)$ parallel along γ with

$$X(2l) = X(0) = X_p \in \dot{\gamma}(0)^{\perp},$$

so that the variation of γ with variation field X satisfies

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(\gamma_s) = -\int R(\dot{\gamma}, X, \dot{\gamma}, X)dt < 0.$$

In other words, $L(\gamma_s) < L(\gamma)$ for all small $s \neq 0$.

We will denote $p_s = \gamma_s(0)$ and let $q_s = \gamma_s(l_s)$ be the point on γ_s which is farthest to p_s. Then

$$\operatorname{dist}(p_s, q_s) < l = \operatorname{inj}(M, g),$$

so there exists a unique normal minimizing geodesic σ_s joint $q_s = \sigma_s(0)$ to p_s. Since $\lim_{s\to 0} q_s = q$, there exists a sequence $s_i \to 0$ so that $\dot{\sigma}_{s_i}(0)$ converges to a unit vector $Y_q \in T_qM$. By continuity, $\sigma(t) = \exp_q(tY_q)$ is a minimizing normal geodesic connecting q to p. In what follows we will show $\dot{\sigma}(0) \perp \dot{\gamma}(l)$, so that σ is not one of the two parts of γ. As a consequence, we get three minimizing geodesic from q to p. This contradicts with propositoin 1.1.

It remains to prove $\dot{\sigma}(0) \perp \dot{\gamma}(l)$. We let $\sigma_{s,t}$ be the minimizing normal geodesic from $p_s = \gamma_s(0)$ to $\gamma_s(t)$ for $\gamma_s(t)$ close to $q_s = \gamma_s(l_s)$. Then $\sigma_{s,t}$ is a variation of $\sigma_s = \sigma_{s,l_s}$. By the choice of q_s, $L(\sigma_{s,t}) \leq L(\sigma_s)$. So according to the first variation formula,

$$0 = \left.\frac{d}{dt}\right|_{t=l_s} E(\sigma_{s,t}) = -\langle \dot{\sigma}_s(0), \dot{\gamma}_s(l_s)\rangle.$$

It follows that $\dot{\sigma}_s(0) \perp \dot{\gamma}_s(l_s)$. Passing to the subsequence s_i and taking limit, we get $\dot{\sigma}(0) = Y_q \perp \dot{\gamma}(l)$. $\square$

Remark. By checking the prove above one can see that for the case $m = \dim M$ even, it's enough to assume that (M, g) is oriented and satisfies the weaker curvature condition $0 < \varepsilon < K \leq 1$ for any ε.

2. The Sphere Theorem

In 1926 Hopf proved that any compact simply connected Riemannian manifold with constant curvature 1 must be the standard round sphere S^m. He conjectured that any compact simply connected Riemannian manifold whose sectional curvature is close to 1 must be homeomorphic to a sphere.

Definition 2.1. Let (M, g) be a Riemannian manifold.

(1) We say (M, g) is *δ-pinched* if $\delta < K \leq 1$.
(2) We say (M, g) is *weakly δ-pinched* if $\delta \leq K \leq 1$.

Example. One can prove that the sectional curvature of the complex projective spaces $\mathbb{CP}^m$ satisfies $\frac{1}{4} \leq K \leq 1$.The same is true for $\mathbb{HP}^m$ and $Ca\mathbb{P}^2$. These spaces are called CROSS (compact rank one symmetric spaces).

Now we are about to prove the following beautiful theorem:

Theorem 2.2 (Topological Sphere Theorem)**.** *Let (M, g) be a complete simply connected Riemannian manifold which is $\frac{1}{4}$-pinched. Then M is homeomorphic to S^m.*

Proof. First by Bonnet-Myers' theorem, M is compact. So there exists $k > \frac{1}{4}$ so that $k \leq K \leq 1$. By Klingenberg's estimate,

$$l := \operatorname{diam}(M, g) \geq \operatorname{inj}(M, g) \geq \pi > \frac{\pi}{2\sqrt{k}}.$$

Take $p, q \in M$ such that $d(p, q) = \operatorname{diam}(M, g)$. Let $q_0 \in M$ be any point in M such that

$$l_1 := d(p, q_0) > \frac{\pi}{2\sqrt{k}},$$

and let γ_1 be a minimizing normal geodesic connecting $p = \gamma_1(0)$ to $q_0 = \gamma_1(l_1)$. According to the following lemma,

Lemma 2.3 (Berger)**.** *Let (M, g) be a compact Riemannian manifold, $p, q \in M$ such that $d(p, q) = \operatorname{diam}(M, g)$. Then for any $X_p \in T_pM$, there exists a minimizing geodesic γ connecting $p = \gamma(0)$ to q so that $\langle \dot{\gamma}(0), X_p \rangle \geq 0$.*

Proof. We will prove this in the next lecture. □

one can find a minimizing normal geodesic γ_2 from $p = \gamma_2(0)$ to $q = \gamma_2(l)$ so that

$$\langle \dot{\gamma}_1(0), \dot{\gamma}_2(0) \rangle \geq 0,$$

i.e. the angle α between $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ is no more than $\frac{\pi}{2}$. Applying the Toponogov comparison theorem to the hinge $\angle q_0pq$, we get

$$d(q_0, q) \leq d(\tilde{q}_0, \tilde{q}),$$

where $\angle \tilde{q}_0\tilde{p}\tilde{q}$ is a comparison hinge in $M_k^m = S^m(\frac{1}{\sqrt{k}})$. By the cosine law in $S^m(\frac{1}{\sqrt{k}})$,

$$\begin{aligned}
\cos(\sqrt{k}d(q_0, q)) &\geq \cos(\sqrt{k}d(\tilde{q}_0, \tilde{q})) \\
&= \cos(\sqrt{k}l)\cos(\sqrt{k}l_1) + \sin(\sqrt{k}l)\sin(\sqrt{k}l_1)\cos(\alpha) \\
&\geq \cos(\sqrt{k}l)\cos(\sqrt{k}l_1) \\
&> 0.
\end{aligned}$$

It follows that $d(q_1, q) < \frac{\pi}{2\sqrt{k}}$. In other words, we proved

$$\overline{B_{\frac{\pi}{2\sqrt{k}}}(p)} \cup \overline{B_{\frac{\pi}{2\sqrt{k}}}(q)} = M.$$

So if we denote

$$r = \frac{1}{2}\left(\mathrm{inj}(M, g) + \frac{\pi}{2\sqrt{k}}\right) > \frac{\pi}{2\sqrt{k}},$$

then

$$M = B_r(p) \cup B_r(q).$$

On the other hand, since $r < \mathrm{inj}(M, g)$, both $B_r(p)$ and $B_r(q)$ are homeomorphic to $\mathbb{R}^m$. Now the sphere theorem follows from the following well-known theorem in topology:

Theorem 2.4 (Brown)**.** *Let M be a smooth compact manifold. If $M = U_1 \cup U_2$, where U_1, U_2 are open subsets in M that are homeomorphic to $\mathbb{R}^m$, then M is homeomorphic to the sphere S^m.*

□

Remark. There is a proof (Tsukamoto, 1962) using only Rauch comparison theorem instead of Toponogov comparison theorem. One can also construct the homeomorphism explicitly instead of using the Brown theorem. Also there is another element proof due to M. Gromov.

Remark. Grove-Shiohahama (1977) proved a generalized sphere theorem where they replace the curvature condition $\frac{1}{4} < K \leq 1$ by the weaker conditions $K \geq \frac{1}{4}$ and $\mathrm{diam}(M, g) > \frac{\pi}{2}$. (We will prove this next time).

Remark. More generally, one has the following rigidity theorem:

Theorem 2.5. *Suppose (M, g) is a complete simply connected Riemannian manifold of dimension m. Then*

(1) *(Berger 1983) If m is even, then there exists $\varepsilon(m) > 0$ so that if $\frac{1}{4} - \varepsilon(m) \leq K \leq 1$, then M is either homeomorphic to S^m or diffeomorphic to one of the CROSSes: $\mathbb{CP}^{m/2}, \mathbb{HP}^{m/4}, \mathrm{Ca}\mathbb{P}^2$.*
(2) *(Abresch-Meyer, 1994) If m is odd, then there exists $\varepsilon > 0$ so that if $\frac{1}{4} - \varepsilon \leq K \leq 1$, then M is homeomorphic to S^m.*

Remark (History of the topological sphere theorem).

- Rauch, 1951: all dimension, $\frac{3}{4}$-pinched.
- Klingenberg 1959: m even and 0.55-pinched.
- Berger 1960: m even and $\frac{1}{4}$-pinched.
- Klingenberg 1961: all dimension, $\frac{1}{4}$-pinched.

Remark (About the differential sphere theorem).

(1) It is natural to ask

$$\boxed{\textbf{Question: } \text{If } M \text{ is } \tfrac{1}{4}\text{-pinched, is } M \text{ diffeomorphic to } S^m?}$$

Note that the problem is totally nontrivial since there exists *exotic spheres*, i.e. manifolds that are homeomorphic to a sphere but not diffeomorphic to a sphere. Whether an exotic sphere admits a Riemannian metric with $K > 0$ is still an open problem. The best known result is due to Wilhelm (2001) who constructed a metric on an exotic sphere with $K > 0$ outside a measure zero set.

(2) [History of the differentiable sphere theorem]
   - Gromoll, Calabi 1966: $h(n)$-pinched, $h(n) \to 1$
   - Sugimoto-Shiohama-Karcher-Ruh 1971: 0.87-pinched.
   - Ruh 1973: 0.80-pinched.
   - Hof-Ruh 1975: $h(n)$-pinched, $h(n) \to 0.68$.
   - Brendle-Schoen 2009: $\frac{1}{4}$-pinched.
   - Peterson-Tao 2010: $\frac{1}{4} - \varepsilon$-pinched $\Longrightarrow$ diffeomorphic to S^m or CROSS

(3) In fact the result of Brendle-Schoen is much stronger, they proved

Theorem 2.6 (Brendle-Schoen 2009)**.** *Let (M, g) be a complete simply connected Riemannian manifold which is pointwise $\frac{1}{4}$-pinched. Then M is diffeomorphic to the sphere S^m.*

Here, *pointwise $\frac{1}{4}$-pinched* means

$$0 < \sup K(\Pi_p) < 4 \inf K](\Pi_p).$$

The pointwise $\frac{1}{4}$-pinched differential sphere theorem for 4-dimensional manifolds was first proven by H. Chen in 1991.

Remark (Sphere theorem in lower dimensions).

(1) For $m = 2$: let M be an oriented compact surface with $K > 0$, then by the Gauss-Bonnet formula
$$0 < \int_M K dA = 2\pi\chi(M).$$
Since the sphere is the only oriented smooth compact surface with positive Euler characteristic ($\chi(S^2) = 2$), we conclude that M is *diffeomorphic* to S^2.
(2) For $m = 3$, by introducing the method of *Ricci flow*, R.Hamilton proved in 1982 that if (M, g) is a 3 dimensional compact Riemannian manifold with $\mathrm{Ric} > 0$, then (M, g) is *diffeomorphic* to S^3.
(3) For $m = 4$ there is a very interesting *conformal sphere theorem*:

Theorem 2.7 (Chang-Gursky-Yang 2003)**.** *Let (M, g) be a compact 4-manifold whose Yamabe invariant is positive. Suppose*
$$\int_M |W|^2 dv < 16\pi^2\chi(M),$$
then M is diffeomorphic to S^4 or $\mathbb{RP}^4$.