LECTURE 24: THE BISHOP-GROMOV VOLUME COMPARISON THEOREM AND ITS APPLICATIONS

1. The Bishop-Gromov Volume Comparison Theorem

Recall that the Riemannian volume density is defined, in an open chart, to be

$$d\text{Vol} = \sqrt{G \circ x^{-1}} dx^1 \cdots dx^m,$$

where $G = \det(g_{ij})$ and $dx^1 \cdots dx^m$ is the Lebesgue measure on $\mathbb{R}^m$.

Now let (M, g) be a complete Riemannian manifold. We have already known (from PSet 3) that the exponential map $\exp_p$ is a diffeomorphism onto the dense open set $M \setminus \text{Cut}(p)$. We denote

$$\Sigma(p) = \exp^{-1}(M \setminus \text{Cut}(p)),$$

then it is an open star-shaped domain in T_pM. We can regard the triple $\{\exp_p^{-1}, M \setminus \text{Cut}(p), \Sigma(p)\}$ as a coordinate chart on M. On $\Sigma(p)$ (or T_pM) we will decompose the Lebesgue measure into via polar coordinates

$$dx^1 \cdots dx^m = r^{m-1} dr d\Theta,$$

where $d\Theta$ is the usual surface measure on S^{m-1}. It follows that there exists a function

$$\mu(r, \Theta) = \sqrt{G \circ \exp_p(r, \Theta)}\, r^{m-1}$$

inside $\Sigma(p)$, so that

$$d\text{Vol} = \mu(r, \Theta) dr d\Theta.$$

For simplicity we will set $\mu(r, \Theta) = 0$ outside $\Sigma(p)$. Note that by definition

$$\overline{B_r(p)} = \exp_p(\overline{B_r(0)}) = \exp_p(\overline{B_r(0)} \cap \overline{\Sigma(p)}\,).$$

Since $\text{Cut}(p)$ is of measure zero in M, we get

$$\text{Vol}(B_r(p)) = \int_{B_r(0) \cap \Sigma(p)} \mu(r, \Theta) dr d\Theta = \int_{B_r(0)} \mu(r, \Theta) dr d\Theta.$$

To calculate the function $\mu(r, \Theta)$, we will fix a direction $\Theta \in S_pM$ and consider Jacobi fields along the geodesic $\gamma(t) = \exp_p(t\Theta)$.

Proposition 1.1. *Let $V_2, \cdots, V_m$ be normal Jacobi fields along the geodesic γ with $V_j(0) = 0$. Suppose $\nabla_{\dot\gamma(0)} V_2, \cdots, \nabla_{\dot\gamma(0)} V_m$ are linearly independent. Then for any point $(r, \Theta) \in \Sigma(p)$,*

$$\mu(r, \Theta) = \frac{\det(V_2(r), \cdots, V_m(r))}{\det(\nabla_{\dot\gamma(0)} V_2, \cdots, \nabla_{\dot\gamma(0)} V_m)},$$

where the determinant is taken with respect to an orthonormal basis in $(\dot\gamma(t))^\perp$.

Proof. Fix any $\Theta \in S_pM$. Let

$$e_1 = \Theta,\ e_2 = \nabla_{\dot\gamma(0)}V_2,\ \cdots,\ e_m = \nabla_{\dot\gamma(0)}V_m.$$

Since they are linearly indepedent, using $e_1, \cdots, e_m$ one can define a set of global linear coordinates $u_1, \cdots, u_m$ on T_pM. Then obviously

$$du_1 \cdots du_m = \frac{r^{m-1}}{\det(e_2, \cdots, e_m)} dr d\Theta.$$

According to lecture 13 (see also lecture 20),

$$V_j(t) = t(d\exp_p)_{t\Theta}(e_j).$$

Note that under $\exp_p$, $u_1, \cdots, u_m$ also gives a coordinate system near $\exp_p(r\Theta)$ if $r\Theta \in \Sigma(p)$. For this coordinate system, we have (at $\exp_p(r\Theta)$)

$$\partial_1 = \dot\gamma(r), \quad \partial_j = \frac{1}{r}V_j(r) \ \ (j \ge 2).$$

It follows

$$r^2 g_{ij}(r,\Theta) = r^2\langle \partial_i, \partial_j\rangle = \langle V_i(r), V_j(r)\rangle$$

for $i, j \ge 2$. Since γ is a normal geodesic, $\langle \partial_1, \partial_1\rangle = 1$. Also by Gauss lemma, $\langle \partial_1, \partial_i\rangle = 0$ for $i \ge 2$. It follows

$$G = \det(g_{ij}) = r^{-2m+2}\det(\langle V_i, V_j\rangle)_{i,j\ge 2} = r^{-2m+2}\det(V_2(r), \cdots, V_m(r))^2.$$

It follows

$$\sqrt{G} du_1 \cdots du_m = \frac{\det(V_2(r), \cdots, V_m(r))}{\det(\nabla_{\dot\gamma(0)}V_2, \cdots, \nabla_{\dot\gamma(0)}V_m)} dr d\Theta.$$

□

Now we compare the volume deisities. In what follows we will denote by $\mu_k(r)$ the function $\mu(r,\Theta)$ for the space form M_k^m (it is obviously independent of p and independent of Θ).

Lemma 1.2. *If* (M, g) *is a complete Riemannian manifold with* Ric $\ge (m-1)k$, *then for any fixed* $\Theta \in S_pM$ *and any* r *with* $r\Theta \in \Sigma(p)$, *we have*

$$\frac{\mu'(r,\Theta)}{\mu(r,\Theta)} \le \frac{\mu_k'(r)}{\mu_k(r)},$$

where the first derivative is taken with respect to r.

Proof. Let γ be the normal geodesic starting at p in the direction Θ. Consider a parallel orthonormal frame $\{e_i(t)\}$ along γ with $e_1(t) = \dot\gamma(t)$. Let $V_j(t)$ be a Jacobi field along γ such that

$$V_j(0) = 0 \quad \text{and} \quad V_j(r) = e_j(r).$$

For simplicity we will denote $A(t) = (\langle V_i(t), V_j(t)\rangle)_{i,j\geq 2}$ and

$$d(t) = \det A(t) = (\det(V_2(r), \cdots, V_m(r)))^2.$$

Then we have $A(r) = \mathrm{Id}$. By matrix algebra (c.f. Lecture 5),

$$d'(t) = d(t)\mathrm{Tr}(A^{-1}(t)A'(t)).$$

So we have

$$\frac{\mu'(r,\Theta)}{\mu(r,\Theta)} = \frac{1}{2}\frac{d'(r)}{d(r)} = \sum_{j=2}^{m}\langle V_j(r), \nabla_{\dot\gamma(r)}V_j(r)\rangle = \sum_{j=2}^{m} I(V_j, V_j).$$

Now consider (c.f. lecture 21)

$$H_j(t) = \frac{sn_k(t)}{sn_k(r)}e_j(t),$$

where $sn_k(t) = \frac{\sin(\sqrt{k}t)}{\sqrt{k}}$ or t or $\frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}}$, depending on whether k is positive, zero or negative. Then H_j and V_j have the same values at end points. So

$$I(V_j, V_j) \leq I(H_j, H_j).$$

Moreover, a direct computation shows

$$\sum_{j=2}^{m} I(H_j, H_j) = \int_0^r \left(\frac{sn_k(t)}{sn_k(r)}\right)^2 [(n-1)k - \mathrm{Ric}(\dot\gamma, \dot\gamma)]dt + \sum_{j=2}^{m}\langle H_j(r), \nabla_{\dot\gamma(r)}H_j\rangle.$$

On the other hand, one can repeat the above computation for the space form M_k^m, and get

$$\frac{\mu_k'(r)}{\mu_k(r)} = \sum_{j=2}^{m}\langle H_j^k(r), \nabla_{\dot\gamma(r)}H_j^k\rangle = \sum_{j=2}^{m}\langle H_j(r), \nabla_{\dot\gamma(r)}H_j\rangle.$$

So we conclude

$$\frac{\mu'(r,\Theta)}{\mu(r,\Theta)} \leq \frac{\mu_k'(r)}{\mu_k(r)}.$$

□

Now we are ready to prove the Bishop-Gromov volume comparison theorem:

Theorem 1.3 (Bishop-Gromov). *If (M, g) is a complete Riemannian manifold with $\mathrm{Ric} \geq (m-1)k$, and $p \in M$ is an arbitrary point. Then the function*

$$r \mapsto \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r^k)}$$

is a non-increasing function which tends to 1 as r goes to 0, where B_r^k is a geodesic ball of radius r in the space form M_k^m. In particular, $\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B_r^k)$.

Proof. For simplicity we will denote

$$a(r) = \int_{S^{m-1}} \mu(r,\Theta)d\Theta, \quad b(r) = \int_{S^{m-1}} \mu_k(r)d\Theta.$$

Then

$$\frac{d}{dr}\left(\log\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r^k)}\right) = \frac{a(r)}{\int_0^r a(t)dt} - \frac{b(r)}{\int_0^r b(t)dt} = \frac{\int_0^r (a(r)b(t)-a(t)b(r))dt}{\left(\int_0^r a(t)dt\right)\left(\int_0^r b(t)dt\right)}.$$

To prove the first part, it suffices to prove $a(r)b(t) - a(t)b(r) \le 0$ for $t \le r$, i.e. $\frac{a(t)}{b(t)}$ is non-increasing. But since $\mu_k(r)$ is independent of Θ,

$$\frac{a(t)}{b(t)} = \int_{S^{m-1}} \frac{\mu(t,\Theta)}{\mu_k(t)}d\Theta.$$

So it suffices to prove that the function $\frac{\mu(t,\Theta)}{\mu_k(t)}$ is non-increasing in t for any fixed Θ. This follows from lemma 1.2 and

$$\frac{d}{dt}\left(\log\frac{\mu(t,\Theta)}{\mu_k(t)}\right) = \frac{\mu'(t,\Theta)}{\mu(t,\Theta)} - \frac{\mu_k'(t)}{\mu_k(t)}.$$

Note that if $t\Theta \notin \Sigma(p)$, then one cannot apply lemma 1.2. However, since $\mu(t,\Theta)=0$ in that case, the function $t \mapsto \frac{\mu(t,\Theta)}{\mu_k(t)}$ is still non-increasing.

The second part follows from theorem 2.5 in lecture 14. □

Remark. By carefully checking the proof of lemma 2.1, one can see that under the assumption of Bishop-Gromov's theorem,

$$\mathrm{Vol}(B_r(p)) = \mathrm{Vol}(B_r^k) \qquad \text{or} \qquad \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r^k)} = \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}(B_R^k)} \quad (r<R)$$

if and only if $B_r(p)$ (or $B_R(p)$ in the second case) is isometric to B_r^k. [Prove this!]

2. Applications of the volume comparison theorem

Let (M,g) be a complete Riemannian manifold with $\mathrm{Ric} \ge 0$. According to Bishop-Gromov volume comparison theorem,

$$\mathrm{Vol}(B_r(p)) \le \mathrm{Vol}(B_r^0) = \omega_m r^m,$$

where ω_m is the volume of unit ball in $\mathbb{R}^m$, with equality holds if and only if (M,g) is isometric with $(\mathbb{R}^m, g_0)$. This gives an upper bound of the volume growth. A natural question is: what is the lower bound of the volume growth? In other words, what is the largest α such that for all r (large enough),

$$\mathrm{Vol}(B_r(p)) \ge cr^\alpha?$$

Of course this question makes sense only for non-compact Riemannian manifolds.

Theorem 2.1 (Calabi-Yau)**.** *Let (M, g) be a complete non-compact Riemannian manifold with $\mathrm{Ric} \geq 0$. Then there exists a positive constant c depending only on p and m so that*

$$\mathrm{Vol}(B_r(p)) \geq cr$$

for any $r > 2$.

Proof. (Following Gromov.) Since M is complete and non-compact, for any $p \in M$ there exists a *ray*, i.e. a geodesic $\gamma : [0, \infty) \to M$ with $\gamma(0) = p$ such that $\mathrm{dist}(p, \gamma(t)) = t$ for all $t > 0$. (Exercise: Prove the existence of a ray.)

For any $t > \frac{3}{2}$, using the Bishop-Gromov volume comparison theorem, we get

$$\frac{\mathrm{Vol}(B_{t+1}(\gamma(t)))}{\mathrm{Vol}(B_{t-1}(\gamma(t)))} \leq \frac{\omega_m (t+1)^m}{\omega_m (t-1)^m} = \frac{(t+1)^m}{(t-1)^m}.$$

On the other hand, by triangle inequality, $B_1(p) \subset B_{t+1}(\gamma(t)) \backslash B_{t-1}(\gamma(t))$. It follows

$$\frac{\mathrm{Vol}(B_1(p))}{\mathrm{Vol}(B_{t-1}(\gamma(t)))} \leq \frac{\mathrm{Vol}\left(B_{t+1}(\gamma(t)) \backslash B_{t-1}(\gamma(t))\right)}{\mathrm{Vol}(B_{t-1}(\gamma(t)))} \leq \frac{(t+1)^m - (t-1)^m}{(t-1)^m},$$

i.e.

$$\mathrm{Vol}(B_{t-1}(\gamma(t))) \geq \mathrm{Vol}(B_1(p)) \frac{(t-1)^m}{(t+1)^m - (t-1)^m} \geq C(m) \mathrm{Vol}(B_1(p)) t,$$

wehre $C(m)$ is the infimum of the function $\frac{1}{t} \frac{(t-1)^m}{(t+1)^m - (t-1)^m}$ on $[\frac{3}{2}, \infty)$, which is positive. Now the theorem follows from the fact

$$B_r(p) \supset B_{\frac{r+1}{2}-1}(\gamma(\frac{r+1}{2})).$$

□

As a second application of volume comparison theorem, we will prove

Theorem 2.2 (S.Y. Cheng)**.** *Let (M, g) be a complete Riemanniian manifold with $\mathrm{Ric} \geq (n-1)k$ for some $k > 0$, and $\mathrm{diam}(M, g) = \frac{\pi}{\sqrt{k}}$, then M is isometric to the standard sphere of radius $\frac{1}{\sqrt{k}}$.*

Proof. (Following Shiohama) For simplicity we may assume $k = 1$.

By Bishop-Gromov volume comparison theorem, for any $p \in M$,

$$\frac{\mathrm{Vol}(B_{\pi/2}(p))}{\mathrm{Vol}(M)} = \frac{\mathrm{Vol}(B_{\pi/2}(p))}{\mathrm{Vol}(B_{\pi}(p))} \geq \frac{\mathrm{Vol}(B^1_{\pi/2})}{\mathrm{Vol}(B^1_{\pi})} = \frac{1}{2}.$$

Now let $p, q \in M$ so that $\mathrm{dist}(p, q) = \pi$. The the above inequality implies

$$\mathrm{Vol}(B_{\pi/2}(p)) \geq \frac{1}{2}\mathrm{Vol}(M), \qquad \mathrm{Vol}(B_{\pi/2}(q)) \geq \frac{1}{2}\mathrm{Vol}(M).$$

Since $B_{\pi/2}(p) \cap B_{\pi/2}(q) = \varnothing$, we must have

$$\frac{\mathrm{Vol}(B_{\pi/2}(p))}{\mathrm{Vol}(B_{\pi}(p))} = \frac{\mathrm{Vol}(B^1_{\pi/2})}{\mathrm{Vol}(B^1_{\pi})} = \frac{1}{2}, \quad \frac{\mathrm{Vol}(B_{\pi/2}(q))}{\mathrm{Vol}(B_{\pi}(q))} = \frac{\mathrm{Vol}(B^1_{\pi/2})}{\mathrm{Vol}(B^1_{\pi})} = \frac{1}{2}.$$

According to Bishop-Gromov comparison theorem, $B_{\pi/2}(p)$ and $B_{\pi/2}(q)$ are both isometric to half sphere. It follows that M is isometric to S^m. □

Finally we will describe some applications of the volume comparison theorem to the fundamental group of complete noncompact Riemannian manifolds with nonnegative Ricci curvatures.

Let's start with some abstract definitions in algebra. Let G be a group. G is said to be *finitely generated* if there exists a finite subset $\Gamma = \{g_1, \cdots, g_N\}$ of G so that any element in G can be represented as group multiplications (or inverse) of elements in Γ. Note that if the group identity element e is in Γ, we can always remove it. Now let's fix a set Γ of generators of G. The *growth function* of G with respect to Γ is defined to be the number of group elements that can be represented as a product of at most k generators, i.e.

$$N_G^\Gamma(k) = \#\{g \in G \mid \exists l \leq k \text{ and } g_{i_1}, \cdots, g_{i_l} \in \Gamma \text{ s.t. } g = g_{i_1}^{\pm 1} \cdots g_{i_l}^{\pm 1}\}.$$

Definition 2.3. Let G be finitely generated, and Γ is a finite set that generates G. We say that G is of (at most) *polynomial growth* if

$$N_G^\Gamma(k) \leq ck^n$$

for some constant c depending only on G, Γ. Similarly we say G is of (at least) *exponential growth* if

$$N_G^\Gamma(k) \geq ce^k.$$

Remark. Note that if Γ' is another finite set of generators, then there exists integers c_1, c_2 so that any element of Γ can be represented via at most c_1 elements of Γ', and any element of Γ' can be represented via at most c_2 elements of Γ. It follows that

$$N_G^\Gamma(k) \geq N_G^{\Gamma'}(c_1 k), \qquad N_G^{\Gamma'}(k) \geq N_G^\Gamma(c_2 k).$$

So the conception of polynomial/exponential growth is independent of the choice of the generating set.

Remark. A celebrated theorem of Gromov: If G is finitely generated and has polynomial growth, then G is virtually nilpotent, i.e. G contains a nilpotent subgroup of finite index.

Coming back to Riemannian manifolds. If (M, g) is a compact Riemannian manifold, and $\widetilde{M}$ its universal covering endowed with the pull-back metric. Then the fundamental group $\pi_1(M)$ acts isometrically on $\widetilde{M}$ as the group of deck transformations. Some known results:

- If M is compact, then $\pi_1(M)$ is finitely generated. [c.f. Munkres, *Topology*, Exercise at the end of chapter 13.][See also Allen Hatcher, *Algebraic Topology*, page 529.]
- (Schwarz 1957) The rate of volume growth of $\widetilde{M}$ is equal to the rate of growth of $\pi_1(M)$.
- (Gromov) If $K \geq 0$, then the set of generates can be choose to be no more than $c(m)$ for some constant c depending only on m. A similar results holds for manifolds with $K \geq -k^2$ and $\mathrm{diam}(M, g) \leq D$. (The proof uses Toporogov comparison theorem. The first part was proved earlier in this course.)
- (Milnor) If $\mathrm{Ric} \geq 0$, then $N_G^\Gamma(k) \leq ck^m$; if $K < 0$, then $N_G^\Gamma(k) \geq ce^k$. (The proof uses volume comparison theorem. See the following theorem for the first part.)

For non-compact Riemannian manifolds, the fundamental group might be not finitely generated in general. However, we have

Theorem 2.4 (Milnor). *Let M be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and let $G \subset \pi_1(M)$ be any finitely generated subgroup. Then there exists a constant c depending only on M and the chosen finite generating set Γ of G so that $N^\Gamma(k) \leq ck^m$.*

Proof. Let Γ be a finite set of generators of G. Fix a point $\tilde{p} \in \widetilde{M}$ and let

$$l = \max\{\mathrm{dist}(\tilde{p}, g_i\tilde{p}) \mid g_i \in \Gamma\}.$$

Then by triangle inequality, for any $g = g_{i_1} \cdots g_{i_k} \in \Gamma^k \subset G$, $\mathrm{dist}(\tilde{p}, g\tilde{p}) \leq kl$. On the other hand side, we can pick

$$\varepsilon = \frac{1}{3}\min\{\mathrm{dist}(\tilde{p}, g\tilde{p}) \mid e \neq g \in G\} > 0$$

so that the balls $B_\varepsilon(g\tilde{p})$ are all disjoint for $g \in G$. It follows

$$B_{kl+\varepsilon}(\tilde{p}) \supset \cup_{g\in\Gamma^k} B_\varepsilon(g\tilde{p})$$

and thus

$$\mathrm{Vol}(B_{kl+\varepsilon}(\tilde{p})) \geq N_G^\Gamma \mathrm{Vol}(B_\varepsilon(p)).$$

Applying the Bishop-Gromov's volume comparison theorem, we get

$$N_G^\Gamma \leq \frac{\mathrm{Vol}(B_{kl+\varepsilon}(\tilde{p}))}{\mathrm{Vol}(B_\varepsilon(p))} \leq \frac{(kl+\varepsilon)^m}{\varepsilon^m} \leq ck^m.$$

□

We end this part of our course by stating the following major conjecture in Riemannian geometry:

Conjecture (Milnor). *Let M be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$, *then $\pi_1(M)$ is finitely generated.*