LECTURE 25: THE HODGE LAPLACIAN

1. The Hodge star operator

Let \((M, g)\) be an oriented Riemannian manifold of dimension \(m\). Then in lecture 3 we have seen that for any orientation-preserving chart, the Riemannian volume form (which is independent of the choice of coordinates) is given by

\[
\omega_g = \sqrt{G} dx^1 \wedge \cdots \wedge dx^m.
\]

Now let \(p \in M\). Then the Riemannian metric \(g\) induces a dual inner product structure on \(T_p^*M\) via

\[
\langle \omega_i dx^i, \eta_j dx^j \rangle = g_{ij} \omega_i \eta_j.
\]

More generally, one can define an inner product on \(\Lambda^k T_p^*M\) as follows: For any orthogonal basis \(\theta^1, \cdots, \theta^m\) of \(T_p^*M\), we require the set

\[
\{ \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \mid i_1 < \cdots < i_k \}
\]

form an orthonormal basis of \(\Lambda^k T_p^*M\). One can check that this definition is independent of the choice of \(\theta^i\)'s. Note that in particular we have

\[
\langle \omega_g, \omega_g \rangle = 1
\]

since in normal coordinates \((g_{ij}) = I\) at \(p\).

As in the case of functions, the pointwise inner product induces an \(L^2\) inner product structure on \(\Omega^k_c(M)\) via

\[
(\omega, \eta) := \int_M \langle \omega, \eta \rangle \omega_g.
\]

To define the Hodge-Laplacian of a differential form, one need to define the so-called Hodge star operator. We first use the pointwise inner product to get an identification between \(\Lambda^k T_p^*M\) and \((\Lambda^k T_p^*M)^*\) that sends \(\beta \in \Lambda^k T_p^*M\) to

\[
L_\beta : \Lambda^k T_p^*M \to \mathbb{R} = \Lambda^m T_p^*M, \quad \alpha \mapsto \langle \alpha, \beta \rangle \omega_g.
\]

On the other hand, the wedge product gives us a non-degenerate pairing

\[
\wedge : \Lambda^k T_p^*M \times \Lambda^{m-k} T_p^*M \to \mathbb{R} = \Lambda^m T_p^*M, \quad (\alpha, \beta) \mapsto \alpha \wedge \beta.
\]

which identifies any element in \((\Lambda^k T_p^*M)^*\) as an element in \(\Lambda^{m-k} T_p^*M\). In particular, for \(\beta \in \Lambda^k T_p^*M\) one can get an element \(\star \beta \in \Lambda^{m-k} T_p^*M\) that is identified with \(L_\beta\), i.e.

\[
\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \omega_g.
\]
This construction gives us a linear isomorphism
\[ \star : \Lambda^k T^*_p M \to \Lambda^{m-k} T^*_p M \]
at each point. Glue these constructions together, we are able to define

**Definition 1.1.** The *Hodge star operator* \( \star : \Omega^k(M) \to \Omega^{m-k}(M) \) maps any \( k \)-form \( \eta \in \Omega^k(M) \) to the \( (m-k) \)-form \( \star \eta \in \Omega^{m-k}(M) \) so that for any \( \omega \in \Omega^k(M) \),
\[ \omega \wedge \star \eta = \langle \omega, \eta \rangle \omega_g. \]

**Remark.** Obviously \( \star \) is \( C^\infty(M) \)-linear.

**Remark.** The \( L^2 \) inner product structure on \( \Omega^k_c(M) \) can be written as
\[ (\omega, \eta) = \int_M \omega \wedge \star \eta. \]

Note that by definition,
\[ \star 1 = \omega_g, \quad \star \omega_g = 1, \quad \omega \wedge \star \eta = \eta \wedge \omega. \]

More generally, if we let \( \omega^1, \cdots, \omega^m \) be a (local) basis of \( T^* M \) so that
\[ \omega^1 \wedge \cdots \wedge \omega^m = f \omega_g \]
is a positive \( m \)-form, then for \( i_1 < \cdots < i_k \), if we let \( j_1 < \cdots < j_{m-k} \) be the complement indeces of \( i \)'s, i.e., such that
\[ \{ i_1, \cdots, i_k, j_1, \cdots, j_{m-k} \} = \{ 1, \cdots, m \}, \]
we have
\[ \star (\omega^{i_1} \wedge \cdots \wedge \omega^{i_k}) = \pm \frac{\langle \omega, \omega \rangle}{f} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{m-k}}, \]
where we denoted \( \omega = \omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \), and the sign \( \pm \) is chosen so that
\[ \omega \wedge \star \omega = \langle \omega, \omega \rangle \omega_g, \]
i.e. so that
\[ \omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \wedge \omega^{j_1} \wedge \cdots \wedge \omega^{j_{m-k}} = \pm \omega^1 \wedge \cdots \wedge \omega^m. \]
One can check that in this case
\[ \pm = (-1)^{i_1+\cdots+i_k+1+\cdots+k}. \]

**Lemma 1.2.** For \( \omega \in \Omega^k(M) \), \( \star \star \omega = (-1)^{k(m-k)} \omega \).

**Proof.** By \( C^\infty(M) \)-linearity, we may assume without loss of generality that
\[ \omega = \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}, \]
where \( \omega^1, \cdots, \omega^m \) is an orthonormal basis at one point \( p \). Then the above computations show
\[ \star \omega = (-1)^{i_1+\cdots+i_k+1+\cdots+k} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{m-k}} \]
and thus
\[ \star \star \omega = (-1)^{j_1+\cdots+j_k+1+\cdots+k}(-1)^{j_1+\cdots+j_{m-k}+1+\cdots+m-k}\omega. \]

It remains to check the following elementary identity
\[ \frac{m(m+1)}{2} + \frac{k(k+1)}{2} + \frac{(m-k)(m-k+1)}{2} \equiv k(m-k) \quad (\text{mod} 2). \]

As a consequence, we see \( \star \) is in fact a linear isometry:

**Corollary 1.3.** For any \( \omega, \eta \in \Omega^k(M) \), one has \( \langle \star \omega, \star \eta \rangle = \langle \omega, \eta \rangle \).

**Proof.** We have
\[ \langle \star \omega, \star \eta \rangle \omega_g = (\star \omega) \wedge (\star \eta) = (-1)^{k(m-k)}(\star \omega) \wedge \eta = \eta \wedge \star \omega = \langle \eta, \omega \rangle \omega_g. \]
So \( \langle \star \omega, \star \eta \rangle = \langle \eta, \omega \rangle = \langle \omega, \eta \rangle \).

**Remark.** The Hodge star operator is of particular importance in dimension 4. In fact, for \( m = 4 \) and \( k = 2 \), the linear map \( \star : \Lambda^2 T^* pM \to \Lambda^2 T^* pM \) satisfies
\[ \star^2 = I. \]

So one can decompose (according to eigenvalues of \( \star \))
\[ \Lambda^2 T^* pM = \Lambda^2_+ T^* pM \oplus \Lambda^2_- T^* pM. \]
Sections of \( \Lambda^2_+ T^* M \) are called self-dual 2-forms, while sections of \( \Lambda^2_- T^* M \) are called anti-self-dual 2-forms.

### 2. The Hodge-Laplace Operator

Using the Hodge star operator, one can define

**Definition 2.1.** The co-differential of \( \omega \in \Omega^k(M) \) is \( \delta \omega \in \Omega^{k-1}(M) \) defined by
\[ \delta \omega = (-1)^{km+m+1} \star d \star \omega. \]

The next lemma states that when we endow all \( \Omega_c^k(M) \)'s with this \( L^2 \) structure, the co-differential operator \( \delta : \Omega_c^k(M) \to \Omega_c^{k-1}(M) \) is the adjoint of the differential operator \( d : \Omega_c^{k-1}(M) \to \Omega_c^k(M) \).

**Lemma 2.2.** For any \( \omega \in \Omega_c^k(M) \) and \( \eta \in \Omega_c^{k-1}(M) \),
\[ \langle \omega, d \eta \rangle = \langle \delta \omega, \eta \rangle. \]

**Proof.** By Stokes' theorem, we have
\[ \langle \omega, d \eta \rangle = (d \eta, \omega) = \int_M d \eta \wedge \star \omega = \int_M d(\eta \wedge \star \omega) - (-1)^{k-1} \eta \wedge d \star \omega = (-1)^k \int_M \eta \wedge d \star \omega \]
On the other hand, by lemma 1.2,
\[ \int_M \eta \wedge (-1)^{m(k+1)+1} \ast d \ast \omega = (-1)^{km+m+1} (-1)^{(m-k+1)(k-1)} \int_M \eta \wedge d \ast \omega, \]
so the conclusion follows from the fact
\[ (-1)^{km+m+1} (-1)^{(m-k+1)(k-1)} = (-1)^k. \]
□

The following formula will be useful.

**Proposition 2.3.** Let \( \{e_i\} \) be an orthonormal frame and \( \{\omega^i\} \) the dual frame. Let \( \nabla \) be the Levi-Civita connection. Then

1. \( d = \omega^i \wedge \nabla_{e_i} \).
2. \( \delta = -\sum_j t_{e_j} \nabla_{e_j} \).

**Proof.**

(1) One can check that the right hand side is independent of choice of basis. So at each point \( p \) that is fixed, with out loss of generality one may take \( e_i = \partial_i \) to be the coordinate vector field for a normal coordinate system centered at \( p \). The dual basis is then \( dx^i \). Recall that by definition, at the point \( p \) one has, for any \( i, j, k \),
\[ (\nabla_{\partial_i} dx^j)(\partial_k) = \nabla_{\partial_i}(dx^j(\partial_k)) - dx^j(\nabla_{\partial_i} \partial_j) = 0. \]
So at \( p \) one has \( \nabla_{\partial_i} dx^j = 0 \) for any \( i, j \).

Now we denote
\[ \bar{d} = \omega^i \wedge \nabla_{e_i} = dx^i \wedge \nabla_{\partial_i}. \]
Consider \( \eta = f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \). Then at \( p \),
\[ \bar{d}\eta = \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = d\eta. \]
This implies \( d = \bar{d} \).

(2) The proof is similar. We denote
\[ \bar{\delta} = -\sum_j t_{e_j} \nabla_{e_j} = -\sum_j t_{\partial_j} \nabla_{\partial_j}. \]
Then at \( p \),
\[ \bar{\delta}\eta = -\sum_j (-1)^{j-1} (\partial_j f) dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_k}. \]
On the other hand, by the definition of \( \delta \) one can calculate \( \delta\eta \) and prove that at \( p \),
\[ \delta\eta = \sum_j (-1)^{j} (\partial_j f) dx^{i_1} \wedge \cdots \wedge dx^{i_j} \wedge \cdots \wedge dx^{i_k}. \]
This completes the proof. □

**Definition 2.4.** The Hodge-Laplace operator on \( k \)-forms is
\[ \Delta = d\delta + \delta d : \Omega^k(M) \to \Omega^k(M). \]
Remark. Since $d^2 = 0$, $\star^2 = \pm 1$, we immediately get
\[ \delta^2 = 0. \]
As a consequence,
\[ \Delta = (d + \delta)^2. \]

Example. One can check that when $k = 0$, the operator $\Delta = \delta d$ equals with the Laplace-Beltrami operator $\Delta$ that we defined in lecture 3. To see this, again we do computation in normal coordinates. Then for any $f \in C^\infty(M)$ we have
\[ df = (\partial_j f) dx^j \]
and thus
\[ \star df = \sum (\partial_j f) (-1)^{j-1} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m, \]
which implies
\[ d \star df = \sum \partial_j (\partial_j f) dx^1 \wedge \cdots \wedge dx^m. \]
It follows
\[ \Delta f = \delta df = - \star d \star df = - \sum \partial_j (\partial_j f), \]
which is exactly the Laplace-Beltrami operator we defined in lecture 3 (but now calculated in normal coordinates).

One can also see this by applying proposition 2.3:
\[ \Delta f = \delta df = - \iota_{e_j} \nabla_e df = - \text{tr}(\nabla^2 f). \]

Like the Beltrami-Laplacian, the Hodge-Laplacian also have very nice propositions:

**Proposition 2.5.** We have

1. $(\omega, \Delta \eta) = (\Delta \omega, \eta)$, i.e. $\Delta$ is symmetric.
2. $(\Delta \omega, \omega) = |\delta \omega|^2 + |d \omega|^2 \geq 0$, i.e. $\Delta$ is non-negative.
3. $\star \Delta = \Delta \star$.

**Proof.** By lemma 2.2, for any $\omega, \eta \in \Omega^k_\mathbb{C}(M),$
\[ (\omega, \Delta \eta) = (\omega, d \delta \eta) + (\omega, \delta d \eta) = (\delta \omega, \delta \eta) + (d \omega, d \eta). \]
Both (1) and (2) follows.

To prove (3), we let $\omega$ be any $k$-form. Then
\[ \star \delta \omega = (-1)^{km+m+1} \star d \star \omega = (-1)^{km+m+1} (-1)^{(m-k+1)(k-1)} d \star \omega = (-1)^k d \star \omega. \]
Similarly
\[ \delta \star \omega = (-1)^{(m-k)m+m+1} d \star \omega = (-1)^{(m-k)m+m+1} (-1)^{k(m-k)} \star d \omega = (-1)^{k+1} \star d \omega. \]
So we get
\[ \star d \delta \omega = (-1)^k \delta \delta \omega = \delta d \star \omega. \]
and

\[ *\delta d \omega = (-1)^{k+1} d * d \omega = d \delta * \omega. \]

It follows

\[ \star \Delta = \star d \delta + \star \delta d = \delta d * \omega + d \delta * = \Delta \star. \]

In problem set 1 we have seen that if \( M \) is connected, then \( \Delta f = 0 \) if and only if \( f \) is a constant function.

**Corollary 2.6.** \( \Delta(f \omega_g) = 0 \) if and only if \( f \) is a constant function.

**Proof.** This follows from

\[ \Delta(f \omega_g) = \Delta * f = \star \Delta f = (\Delta f) \omega_g. \]

**Definition 2.7.** A \( k \)-form \( \omega \) is called harmonic if \( \Delta \omega = 0 \).

We will denote the set of all harmonic \( k \)-forms on \((M, g)\) by \( \mathcal{H}^k(M) \). It is obviously a vector space. Obviously if \( M \) is connected,

\[ \mathcal{H}^0(M) \simeq \mathbb{R}, \quad \mathcal{H}^m(M) \simeq \mathbb{R}. \]

According to proposition 2.3, if \( \omega \in \Omega^k(M) \) is parallel, i.e. \( \nabla \omega = 0 \), then \( \omega \) is harmonic.

**Example.** Consider \( M = \mathbb{T}^m \) equipped with the standard flat metric. Then any \( k \)-form can be written as

\[ \omega = \sum \omega_{i_1 \ldots i_m} dx^{i_1} \land \cdots \land dx^{i_m}. \]

It is not hard to see that each \( dx^{i_1} \land \cdots \land dx^{i_m} \) is parallel. So one can see \( \Delta \omega = 0 \) if and only if \( \Delta \omega_{i_1 \ldots i_k} = 0 \). As a consequence, we see

\[ \dim \mathcal{H}^k(\mathbb{T}^m) = \binom{n}{k}. \]

The following proposition can be viewed as an alternative definition of harmonic forms: [For example, in symplectic Hodge theory, there is no \( \Delta \), however, one can still define harmonic form by this method.]

**Corollary 2.8.** Suppose \( M \) is closed. Then

\[ \omega \in \mathcal{H}^k(M) \iff d \omega = 0, \delta \omega = 0. \]

**Proof.** If \( \Delta \omega = 0 \), then by proposition 2.5, one must have \( d \omega = 0, \delta \omega = 0 \).

Conversely if \( d \omega = 0, \delta \omega = 0 \), then of course \( \Delta \omega = 0 \). \( \square \)

**Corollary 2.9.** If \( \omega \in \mathcal{H}^k(M) \), then \( \star \omega \in \mathcal{H}^{m-k}(M) \).

**Proof.** \( \Delta \star \omega = \star \Delta \omega = 0 \). \( \square \)