LECTURE 25: THE HODGE LAPLACIAN

1. The Hodge star operator

Let (M, g) be an oriented Riemannian manifold of dimension m. Then in lecture 3 we have seen that for any orientation-preserving chart, the Riemannian volume form (which is independent of the choice of coordinates) is given by

$$\omega_g = \sqrt{G}dx^1 \wedge \cdots \wedge dx^m.$$

Now let $p \in M$. Then the Riemannian metric g induces a dual inner product structure on T_p^*M via

$$\langle \omega_i dx^i, \eta_j dx^j \rangle = g^{ij}\omega_i \eta_j.$$

More generally, one can define an inner product on $\Lambda^k T_p^* M$ as follows: For any orthogonal basis $\theta^1, \cdots, \theta^m$ of T_p^*M, we require the set

$$\{\theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \mid i_1 < \cdots < i_k\}$$

form an orthonormal basis of $\Lambda^k T_p^* M$. One can check that this definition is independent of the choice of θ^i's. Note that in particular we have

$$\langle \omega_g, \omega_g \rangle = 1$$

since in normal coordinates $(g_{ij}) = I$ at p.

As in the case of functions, the pointwise inner product induces an L^2 inner product structure on $\Omega_c^k(M)$ via

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle \omega_g.$$

To define the Hodge-Laplacian of a differential form, one need to define the so-called Hodge star operator. We first use the pointwise inner product to get an identification between $\Lambda^k T_p^* M$ and $(\Lambda^k T_p^* M)^*$ that sends $\beta \in \Lambda^k T_p^* M$ to

$$L_\beta : \Lambda^k T_p^* M \to \mathbb{R} = \Lambda^m T_p^* M, \quad \alpha \mapsto \langle \alpha, \beta \rangle \omega_g.$$

On the other hand, the wedge product gives us a non-degenerate pairing

$$\wedge : \Lambda^k T_p^* M \times \Lambda^{m-k} T_p^* M \to \mathbb{R} = \Lambda^m T_p^* M, \quad (\alpha, \beta) \mapsto \alpha \wedge \beta.$$

which identifies any element in $(\Lambda^k T_p^* M)^*$ as an element in $\Lambda^{m-k} T_p^* M$. In particular, for $\beta \in \Lambda^k T_p^* M$ one can get an element $\star\beta \in \Lambda^{m-k} T_p^* M$ that is identified with L_β, i.e.

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega_g.$$

This construction gives us a linear isomorphism

$$\star : \Lambda^k T_p^* M \to \Lambda^{m-k} T_p^* M$$

at each point. Glue these constructions together, we are able to define

Definition 1.1. The *Hodge star operator* $\star : \Omega^k(M) \to \Omega^{m-k}(M)$ maps any k-form $\eta \in \Omega^k(M)$ to the $(m-k)$-form $\star\eta \in \Omega^{m-k}(M)$ so that for any $\omega \in \Omega^k(M)$,

$$\omega \wedge \star\eta = \langle \omega, \eta \rangle \omega_g.$$

Remark. Obviously $\star$ is $C^\infty(M)$-linear.

Remark. The L^2 inner product structure on $\Omega_c^k(M)$ can be written as

$$(\omega, \eta) = \int_M \omega \wedge \star\eta.$$

Note that by definition,

$$\star 1 = \omega_g, \quad \star\omega_g = 1, \quad \omega \wedge \star\eta = \eta \wedge \star\omega.$$

More generally, if we let $\omega^1, \cdots, \omega^m$ be a (local) basis of T^*M so that

$$\omega^1 \wedge \cdots \wedge \omega^m = f\omega_g$$

is a positive m-form, then for $i_1 < \cdots < i_k$, if we let $j_1 < \cdots < j_{m-k}$ be the complement indeces of i's, i.e., such that

$$\{i_1, \cdots, i_k, j_1, \cdots, j_{m-k}\} = \{1, \cdots, m\},$$

we have

$$\star(\omega^{i_1} \wedge \cdots \wedge \omega^{i_k}) = \pm \frac{\langle \omega, \omega \rangle}{f} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{m-k}},$$

where we denoted $\omega = \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}$, and the sign $\pm$ is chosen so that

$$\omega \wedge \star\omega = \langle \omega, \omega \rangle \omega_g,$$

i.e. so that

$$\omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \wedge \omega^{j_1} \wedge \cdots \wedge \omega^{j_{m-k}} = \pm \omega^1 \wedge \cdots \wedge \omega^m.$$

One can check that in this case

$$\pm = (-1)^{i_1 + \cdots + i_k + 1 + \cdots + k}.$$

Lemma 1.2. *For $\omega \in \Omega^k(M)$, $\star\star\omega = (-1)^{k(m-k)}\omega$.*

Proof. By $C^\infty(M)$-linearity, we may assume without loss of generality that

$$\omega = \omega^{i_1} \wedge \cdots \wedge \omega^{i_k},$$

where $\omega^1, \cdots, \omega^m$ is an orthonormal basis at one point p. Then the above computations show

$$\star\omega = (-1)^{i_1 + \cdots + i_k + 1 + \cdots + k} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{m-k}}$$

and thus
$$\star\star\omega = (-1)^{i_1+\cdots+i_k+1+\cdots+k}(-1)^{j_1+\cdots+j_{m-k}+1+\cdots+m-k}\omega.$$
It remains to check the following elementary identity
$$\frac{m(m+1)}{2}+\frac{k(k+1)}{2}+\frac{(m-k)(m-k+1)}{2}\equiv k(m-k) \qquad (\mathrm{mod} 2).$$
□

As a consequence, we see $\star$ is in fact a linear isometry:

Corollary 1.3. *For any $\omega, \eta \in \Omega^k(M)$, one has $\langle \star\omega, \star\eta\rangle = \langle \omega, \eta\rangle$.*

Proof. We have
$$\langle \star\omega, \star\eta\rangle\omega_g = (\star\omega)\wedge\star(\star\eta) = (-1)^{k(m-k)}(\star\omega)\wedge\eta = \eta\wedge\star\omega = \langle \eta, \omega\rangle\omega_g.$$
So $\langle \star\omega, \star\eta\rangle = \langle \eta, \omega\rangle = \langle \omega, \eta\rangle$. □

Remark. The Hodge star operator is of particular important in dimension 4. In fact, for $m = 4$ and $k = 2$, the linear map $\star : \Lambda^2 T_p^*M \to \Lambda^2 T_p^*M$ satisfies
$$\star^2 = I.$$
So one can decompose (according to eigenvalues of $\star$)
$$\Lambda^2 T_p^*M = \Lambda^2_+ T_p^*M \oplus \Lambda^2_- T_p^*M.$$
Sections of $\Lambda^2_+ T^*M$ are called *self-dual* 2-forms, while sections of $\Lambda^2_- T^*M$ are called *anti-self-dual* 2-forms.

2. The Hodge-Laplace operator

Using the Hodge star operator, one can define

Definition 2.1. The *co-differential* of $\omega \in \Omega^k(M)$ is $\delta\omega \in \Omega^{k-1}(M)$ defined by
$$\delta\omega = (-1)^{km+m+1} \star d \star.$$

The next lemma states that when we endow all $\Omega^k_c(M)$'s with this L^2 structure, the co-differential operator $\delta : \Omega^k_c(M) \to \Omega^{k-1}_c(M)$ is the *adjoint* of the differential operator $d : \Omega^{k-1}_c(M) \to \Omega^k_c(M)$.

Lemma 2.2. *For any $\omega \in \Omega^k_c(M)$ and $\eta \in \Omega^{k-1}_c(M)$,*
$$(\omega, d\eta) = (\delta\omega, \eta).$$

Proof. By Stokes' theorem, we have
$$(\omega, d\eta) = (d\eta, \omega) = \int_M d\eta\wedge\star\omega = \int_M d(\eta\wedge\star\omega) - (-1)^{k-1}\eta\wedge d\star\omega = (-1)^k\int_M \eta\wedge d\star\omega$$

On the other hand, by lemma 1.2,

$$((-1)^{km+m+1}\star d\star\omega,\eta)=\int_M \eta\wedge(-1)^{m(k+1)+1}\star\star d\star\omega=(-1)^{km+m+1}(-1)^{(m-k+1)(k-1)}\int_M \eta\wedge d\star\omega,$$

so the conclusion follows from the fact $(-1)^{km+m+1}(-1)^{(m-k+1)(k-1)}=(-1)^k$. □

The following formula will be useful.

Proposition 2.3. *Let* $\{e_i\}$ *be an orthonormal frame and* $\{\omega^i\}$ *the dual frame. Let* ∇ *be the Levi-Civita connection. Then*

(1) $d=\omega^i\wedge\nabla_{e_i}$.
(2) $\delta=-\sum_j \iota_{e_j}\nabla_{e_j}$.

Proof. (1) One can check that the right hand side is independent of choice of basis. So at each point p that is fixed, with out loss of generality one may take $e_i=\partial_i$ to be the coordinate vector field for a normal coordinate system centered at p. The dual basis is then dx^i. Recall that by definition, at the point p one has, for any i,j,k,

$$(\nabla_{\partial_i}dx^j)(\partial_k)=\nabla_{\partial_i}(dx^j(\partial_k))-dx^j(\nabla_{\partial_i}\partial_j)=0.$$

So at p one has $\nabla_{\partial_i}dx^j=0$ for any i,j.

Now we denote

$$\bar d=\omega^i\wedge\nabla_{e_i}=dx^i\wedge\nabla_{\partial_i}.$$

Consider $\eta=f dx^{i_1}\wedge\cdots\wedge dx^{i_k}$. Then at p,

$$\bar d\eta=\frac{\partial f}{\partial x^i}dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}=d\eta.$$

This implies $d=\bar d$.

(2) The proof is similar. We denote

$$\bar\delta=-\sum_j \iota_{e_j}\nabla_{e_j}=-\sum_j \iota_{\partial_j}\nabla_{\partial_j}.$$

Then at p,

$$\bar\delta\eta=-\sum_j(-1)^{j-1}(\partial_{i_j}f)dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_j}}\wedge\cdots\wedge dx^{i_k}.$$

On the other hand, by the definition of δ one can calculate $\delta\eta$ and prove that at p,

$$\delta\eta=\sum_j(-1)^{j}(\partial_{i_j}f)dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_j}}\wedge\cdots\wedge dx^{i_k}.$$

This completes the proof. □

Definition 2.4. The *Hodge-Laplace operator* on k-forms is

$$\Delta=d\delta+\delta d:\Omega^k(M)\to\Omega^k(M).$$

Remark. Since $d^2 = 0, \star^2 = \pm 1$, we immediately get

$$\delta^2 = 0.$$

As a consequence,

$$\Delta = (d+\delta)^2.$$

Example. One can check that when $k = 0$, the operator $\Delta = \delta d$ equals with the Laplace-Beltrami operator Δ that we defined in lecture 3. To see this, again we do computation in normal coordinates. Then for any $f \in C^\infty(M)$ we have

$$df = (\partial_j f) dx^j$$

and thus

$$\star df = \sum (\partial_j f)(-1)^{j-1} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m,$$

which implies

$$d \star df = \sum \partial_j (\partial_j f) dx^1 \wedge \cdots \wedge dx^m.$$

It follows

$$\Delta f = \delta df = - \star d \star df = -\sum \partial_j(\partial_j f),$$

which is exactly the Laplace-Beltrami operator we defined in lecture 3 (but now calculated in normal coordinates).

One can also see this by applying proposition 2.3:

$$\Delta f = \delta df = -\iota_{e_j} \nabla_{e_j} df = -\mathrm{tr}(\nabla^2 f).$$

Like the Beltrami-Laplacian, the Hodge-Laplacian also have very nice propositions:

Proposition 2.5. *We have*

1. $(\omega, \Delta\eta) = (\Delta\omega, \eta)$, *i.e.* Δ *is symmetric.*
2. $(\Delta\omega, \omega) = |\delta\omega|^2 + |d\omega|^2 \geq 0$, *i.e.* Δ *is non-negative.*
3. $\star\Delta = \Delta\star$.

Proof. By lemma 2.2, for any $\omega, \eta \in \Omega_c^k(M)$,

$$(\omega, \Delta\eta) = (\omega, d\delta\eta) + (\omega, \delta d\eta) = (\delta\omega, \delta\eta) + (d\omega, d\eta).$$

Both (1) and (2) follows.

To prove (3), we let ω be any k-form. Then

$$\star\delta\omega = (-1)^{km+m+1} \star \star d \star \omega = (-1)^{km+m+1}(-1)^{(m-k+1)(k-1)} d \star \omega = (-1)^k d \star \omega.$$

Similarly

$$\delta \star \omega = (-1)^{(m-k)m+m+1} \star d \star \star\omega = (-1)^{(m-k)m+m+1}(-1)^{k(m-k)} \star d\omega = (-1)^{k+1} \star d\omega.$$

So we get

$$\star d\delta\omega = (-1)^k \delta \star \delta\omega = \delta d \star \omega$$

and

$$\star\delta d\omega = (-1)^{k+1} d \star d\omega = d\delta \star \omega.$$

It follows

$$\star\Delta = \star d\delta + \star\delta d = \delta d \star \omega + d\delta\star = \Delta \star .$$

□

In problem set 1 we have seen that if M is connected, then $\Delta f = 0$ if and only if f is a constant function.

Corollary 2.6. $\Delta(f\omega_g) = 0$ *if and only if* f *is a constant function.*

Proof. This follows from

$$\Delta(f\omega_g) = \Delta \star f = \star\Delta f = (\Delta f)\omega_g.$$

□

Definition 2.7. A k-form ω is called *harmonic* if $\Delta\omega = 0$.

We will denote the set of all harmonic k-forms on (M, g) by $\mathcal{H}^k(M)$. It is obviously a vector space. Obviously if M is connected,

$$\mathcal{H}^0(M) \simeq \mathbb{R}, \quad \mathcal{H}^m(M) \simeq \mathbb{R}.$$

According to proposition 2.3, if $\omega \in \Omega^k(M)$ is parallel, i.e. $\nabla\omega = 0$, then ω is harmonic.

Example. Consider $M = \mathbb{T}^m$ equipped with the standard flat metric. Then any k-form can be written as

$$\omega = \sum \omega_{i_1 \cdots i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m}.$$

It is not hard to see that each $dx^{i_1} \wedge \cdots \wedge dx^{i_m}$ is parallel. So one can see $\Delta\omega = 0$ if and only if $\Delta\omega_{i_1 \cdots i_k} = 0$. As a consequence, we see

$$\dim \mathcal{H}^k(T^m) = \binom{n}{k}.$$

The following proposition can be viewed as an alternative definition of harmonic forms: [For example, in symplectic Hodge theory, there is no Δ, however, one can still define harmonic form by this method.]

Corollary 2.8. *Suppose* M *is closed. Then*

$$\omega \in \mathcal{H}^k(M) \Longleftrightarrow d\omega = 0, \delta\omega = 0.$$

Proof. If $\Delta\omega = 0$, then by proposition 2.5, one must have $d\omega = 0, \delta\omega = 0$.

Conversely if $d\omega = 0, \delta\omega = 0$, then of course $\Delta\omega = 0$. □

Corollary 2.9. *If* $\omega \in \mathcal{H}^k(M)$*, then* $\star\omega \in \mathcal{H}^{m-k}(M)$*.*

Proof. $\Delta \star \omega = \star\Delta\omega = 0.$ □