LECTURE 27: THE BOCHNER TECHNIQUE

1. Bochner's Technique

Let ω be a k-form. One can think of ω as a $(0,k)$-tensor. Recall $\nabla\omega$ is then a $(0,k+1)$ tensor, and $\nabla^2\omega$ is a $(0,k+2)$ tensor. Let $\{e_i\}$ be an orthonormal frame. In what follows we will always denote $\{\omega^i\}$ the dual coframe of $\{e_i\}$. We define

$$\mathrm{tr}(\nabla^2\omega)(\cdots) := \sum(\nabla^2\omega)(\cdots, e_i, e_i).$$

Obviously the definition is independent of the choice of an orthonormal frame $\{e_i\}$.

Lemma 1.1. $\mathrm{tr}(\nabla^2\omega) = \sum\left(\nabla_{e_i}\nabla_{e_i}\omega - \nabla_{\nabla_{e_i}e_i}\omega\right)$.

Proof. One can check the right hand side is also independent of the choice of orthonormal frames [In the computation we see $\nabla_{e_i}\nabla_{e_i}\omega$ depends on the choice of orthonormal frame]: If we let $\{f_i\}$ be another orthonormal frame, then $e_i = c_i^j f_j$ for some orthogonal matrix (c_i^j). So $\sum_i c_i^j c_i^k = \delta^{jk}$ and thus

$$\begin{aligned}\sum_i \nabla_{e_i}\nabla_{e_i}\omega &= \sum_{i,j,k}\nabla_{c_i^j f_j}\nabla_{c_i^k f_k}\omega \\ &= \sum_{i,j,k} c_i^j f_j(c_i^k)\nabla_{f_k}\omega + \sum_k \nabla_{f_k}\nabla_{f_k}\omega.\end{aligned}$$

Similarly one can check

$$\sum_i \nabla_{\nabla_{e_i}e_i}\omega = \sum_{i,j,k} c_i^j f_j(c_i^k)\nabla_{f_k}\omega + \sum_k \nabla_{\nabla_{f_k}f_k}\omega.$$

So to prove the lemma, it is enough to prove the formula in a particular frame. To make the computation as easy as possible, we will use the normal frame around a point p. Then at p, $\nabla_{e_i}e_j = 0$ for all i,j. In the following computations we will choose $\cdots$ to be $e_j's$. Then at p,

$$\begin{aligned}\nabla^2\omega(\cdots, e_i, e_i) &= (\nabla_{e_i}\nabla\omega)(\cdots, e_i) \\ &= e_i\left((\nabla\omega)(\cdots, e_i)\right) - (\nabla\omega)(\cdot, \nabla_{e_i}\cdot, \cdot, e_i) \\ &= e_i\left((\nabla\omega)(\cdots, e_i)\right) \\ &= e_i((\nabla_{e_i}\omega)(\cdots)).\end{aligned}$$

On the other hand, at p we also have

$$(\nabla_{e_i}\nabla_{e_i}\omega)(\cdots) = e_i((\nabla_{e_i}\omega)(\cdots)) - (\nabla_{e_i}\omega)(\cdot, \nabla_{e_i}\cdot, \cdot) = e_i((\nabla_{e_i}\omega)(\cdots)).$$

So the formula follows. $\square$

Theorem 1.2 (Weitzenböck formula). *For any k-form ω,*

$$\Delta\omega = -\mathrm{tr}(\nabla^2\omega) + \omega^i \wedge \iota_{e_j} R(e_i, e_j)\omega.$$

Proof. Similarly one can check that the right hand side is independent of the choice of orthonormal frame. So we only need to do the computations at one point p using a normal frame $\{e_i\}$ centered at p. Since $\nabla_{e_i} e_j = 0$, we get (c.f. Lecture 5)

$$\nabla_{e_i}(C(e_j \otimes \nabla_{e_j}\omega)) = C(\nabla_{e_i}(e_j \otimes \nabla_{e_j}\omega)) = C(e_j \otimes \nabla e_i \nabla_{e_j}\omega).$$

Recall

$$d = \omega^i \wedge \nabla_{e_i}, \quad \delta = -\sum_j \iota_{e_j}\nabla_{e_j},$$

so together with the fact $\nabla_{e_j}\omega^i = 0$ we get

$$\begin{aligned}
\Delta\omega &= d\delta\omega + \delta d\omega \\
&= -\omega^i \wedge \nabla_{e_i}(\iota_{e_j}\nabla_{e_j}\omega) - \iota_{e_j}\nabla_{e_j}(\omega^i \wedge \nabla_{e_i}\omega) \\
&= -\omega^i \wedge \iota_{e_j}\nabla_{e_i}\nabla_{e_j}\omega - \iota_{e_j}(\omega^i \wedge \nabla_{e_j}\nabla_{e_i}\omega) \\
&= -\omega^i \wedge \iota_{e_j}\nabla_{e_i}\nabla_{e_j}\omega - \nabla_{e_i}\nabla_{e_i}\omega + \omega^i \wedge \iota_{e_j}\nabla_{e_j}\nabla_{e_i}\omega \\
&= -\mathrm{tr}(\nabla^2\omega) + \omega^i \wedge \iota_{e_j} R(e_i, e_j)\omega.
\end{aligned}$$

□

As a consequence, we have

Proposition 1.3 (Bochner). *For any k-form,*

$$-\frac{1}{2}\Delta|\omega|^2 = -\langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + F(\omega),$$

where $F(\omega) = \langle \omega^i \wedge \iota_{e_j} R(e_i, e_j)\omega, \omega\rangle$.

Proof. Again let's do computation in local normal coordinates. We have

$$\begin{aligned}
-\langle \Delta\omega, \omega\rangle + F(\omega) = \langle \mathrm{tr}(\nabla^2\omega), \omega\rangle &= \left\langle \sum \nabla_{e_i}\nabla_{e_i}\omega, \omega \right\rangle \\
&= \sum \nabla_{e_i}\langle \nabla_{e_i}\omega, \omega\rangle - \langle \nabla_{e_i}\omega, \nabla_{e_i}\omega\rangle \\
&= \frac{1}{2}\sum \nabla_{e_i}\nabla_{e_i}(\langle \omega, \omega\rangle|) - |\nabla\omega|^2 \\
&= -\frac{1}{2}\Delta|\omega|^2 - |\nabla\omega|^2.
\end{aligned}$$

□

Now suppose $k = 1$, i.e. ω is a 1-form. Then

$$\langle (R(X,Y)\omega)(Z) = [(-\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]})\omega](Z) = -\omega(R(X,Y)Z)$$

since

$$\begin{aligned}(\nabla_X\nabla_Y\omega)(Z) &= X((\nabla_Y\omega)(Z)) - (\nabla_Y\omega)(\nabla_X Z)\\ &= X(Y(\omega(Z))) - X(\omega(\nabla_Y Z)) - Y(\omega(\nabla_X Z)) + \omega(\nabla_Y\nabla_X Z)\end{aligned}$$

and

$$(\nabla_{[X,Y]}\omega)(Z) = [X,Y](\omega(Z)) - \omega(\nabla_{[X,Y]}Z).$$

So if we denote $\sharp\omega$ be the vector field corresponds to ω that we introduced in lecture 1, then with respect to orthonormal frame, $\sharp\omega = \langle\omega,\omega^i\rangle e_i$. So

$$\begin{aligned}F(\omega) = \langle\omega^i\wedge\iota_{e_j}R(e_i,e_j)\omega,\omega\rangle &= \left(\iota_{e_j}R(e_i,e_j)\omega\right)\langle\omega^i,\omega\rangle\\ &= -\omega(R(e_i,e_j)e_j)\langle\omega^i,\omega\rangle\\ &= -\langle\sharp\omega, R(e_i,e_j)e_j\rangle\langle\omega^i,\omega\rangle\\ &= -\langle\sharp\omega, R(\sharp\omega,e_j)e_j\rangle\\ &= Ric(\sharp\omega,\sharp\omega).\end{aligned}$$

In other words, we get

Corollary 1.4. *For any 1-form ω,*

$$-\frac{1}{2}\Delta|\omega|^2 = -\langle\Delta\omega,\omega\rangle + |\nabla\omega|^2 + Ric(\sharp\omega,\sharp\omega).$$

Now we are ready to prove

Theorem 1.5 (Bochner)**.** *Let (M,g) be a closed oriented RIemannian manifold.*

(1) *If $Ric \ge 0$ on M, then any harmonic 1-form ω is parallel, i.e. $\nabla\omega = 0$.*
(2) *If $Ric \ge 0$ on M but $Ric > 0$ at one point, then there is no non-trivial harmonic 1-form.*

Proof. Recall from lecture 3 that for any function f, $\int_M \Delta f dx = 0$. Apply this to $f = |\omega|^2$, where ω is a harmonic 1-form on M, we get

$$0 = \int_M \left(|\nabla\omega|^2 + Ric(\sharp\omega,\sharp\omega)\right) dx \ge 0.$$

So if $Ric \ge 0$ we must have $\nabla\omega = 0$, i.e. ω is parallel; if $Ric > 0$ we must have $\sharp\omega = 0$, i.e. $\omega = 0$. □

Combine the result with Hodge theorem $\mathcal{H}^k \simeq H^k_{dR}(M)$, we immediately get the following theorem on the Betti number $b_1 = \dim H^1_{dR}(M)$:

Corollary 1.6. *Let (M,g) be a closed oriented Riemannian manifold.*

(1) *If $Ric \ge 0$ on M, then $b_1(M) \le \dim M$.*
(2) *If $Ric \ge 0$ on M but $Ric > 0$ at one point, then $b_1(M) = 0$.*

Proof. Case (1) follows from the fact that any parallel 1-form is determined by its value at one point. Case (2) is clear. □

Remark. For higher Betti number,

- The curvature operator $\mathcal{R} : \Lambda^2 TM \to \Lambda^2 TM$ is defined as

  $$\mathcal{R}(e_i \wedge e_j) = R_{ijkl} e_k \wedge e_l,$$

  where $\{e_i\}$ is a local orthonormal frame. One can check that the definition is independent of the choice of $\{e_i\}$'s. Moreover, it is symmetric:

  $$\langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle = \langle X \wedge Y, \mathcal{R}(Z \wedge W) \rangle.$$

  So the eigenvalues of $\mathcal{R}$ are real numbers. We say (M, g) has *nonnegative (positive) curvature operator* if all eigenvalues of $\mathcal{R}$ are nonnegative (positive). Moreover, one can show that if $\mathcal{R}$ is nonnegative (positive), then the sectional curvature is nonnegative (positive). Similarly one can prove

  Theorem 1.7. *Let (M, g) be a closed oriented Riemannian manifold.*
  (1) *If $\mathcal{R}$ is nonnegative, then any harmonic k-form is parallel. In particular, $b_k(M) \le \binom{m}{k}$.*
  (2) *If $\mathcal{R}$ is nonnegative and is postive at one point, then there is no nontrivial harmonic k-form. In particular, $b_k(M) = 0$ for $0 < k < m$.*

- People conjectured that if $K > 0$, then $b_2(M) \le 1$. Note that this conjecture implies Hopf's problem.

For vector fields, we have a similar formula:

Theorem 1.8. *Let X be a vector field so that $\flat X$ is a closed 1-form,*

$$-\frac{1}{2}\Delta |X|^2 = |\nabla X|^2 + \langle \nabla(\mathrm{div} X), X \rangle + Ric(X, X).$$

Proof. One just apply corollary 1.4 to $\omega = \flat X$. It is not hard to check $|\omega|^2 = |X|^2$ and $|\nabla \omega|^2 = |\nabla X|^2$. So the formula follows from

$$\langle \Delta(\flat X), \flat X \rangle = \langle d\delta(\flat X), \flat X \rangle = \langle \nabla \delta(\flat X), X \rangle$$

and the fact [Prove this!] $\mathrm{div} X = -\delta(\flat X)$. □

For any smooth function f on M, the gradient field ∇f satisfies the condition since $\flat(\nabla f) = df$. So one gets

Corollary 1.9. *For any smooth function f,*

$$-\frac{1}{2}\Delta |\nabla f|^2 = |\nabla^2 f|^2 - \langle \nabla(\Delta f), \nabla f \rangle + Ric(\nabla f, \nabla f).$$

2. Eigenvalues of the Laplacian

Let (M, g) be a closed oriented Riemannian manifold. We call a number λ an *eigenvalue* of Δ, if there exists a smooth function $u \neq 0$ so that

$$\Delta u = \lambda u.$$

We have seen from PSet 1 that

- All eigenvalues of Δ are non-negative real numbers.
- $\lambda = 0$ is always an eigenvalue, whose eigenfunctions are constant functions.
- If u_1 and u_2 are eigenfunctions of different eigenvalues, then $u_1 \perp u_2$.

Obviously the first and the third fact can easily be extended to the Hodge Laplcian, and the second fact should be modified: the eigenspace that corresponds to the eigenvalue 0 is exactly the space $\mathcal{H}$ of harmonic k-forms.

Now with Hodge decomposition we can say more on the eigenvalues. Last time we have seen that Δ is invertible on $\mathcal{H}^{\perp}$, and the inverse G is a compact operator. So according to the structural theory of the spectrum of compact operators, we see that the eigenvalues of G (restricted to $\overline{\mathcal{H}^{\perp}}$) must be a sequence of postive numbers that tends to 0, each eigenvalue is of finite multiplicity, and 0 is the only limit point. Moreover, the eigenfunctions can be chosen to be an orthonormal basis of the whole space. As a consequence, we get

Theorem 2.1. *The eigenvalues of Δ form an increasing sequence that tends to ∞:*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \to \infty.$$

Moreover, each eigenvalue has finite multiplicity, and one can choose an eigenbasis $\{u_1, u_2, u_3, \cdots\}$ which form a complete orthonormal basis of $L^2(M)$.

For the case of functions, i.e. the Laplace-Beltrami operator, since 0 is always an eigenvalue of multiplicity 1, one usually denote the eigenvalue sequence as

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \to \infty.$$

The first non-zero eigenvalue λ_1 is quite interesting. Finally let's apply Bochner's formula to prove a lower bound estimate (and a rigidity theorem) for λ_1.

Theorem 2.2 (Lichnerowitz)**.** *Let (M, g) be a closed Riemannian manifold with $Ric \geq (m-1)C$ for some $C > 0$. Then the first eigenvalue*

$$\lambda_1 \geq mC.$$

Proof. First by Schwartz inequality, for any function f we have

$$|\nabla^2 f|^2 \geq \frac{1}{m}(tr(\nabla^2 f))^2 = \frac{1}{m}(\Delta f)^2.$$

So if we take $f = u$ be an eigenfunction, i.e. $\Delta u = \lambda u$, and apply corollary 1.9, we get

$$-\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{\lambda}{m}u\Delta u - \lambda\langle\nabla u, \nabla u\rangle + Ric(\nabla u, \nabla u). \tag{1}$$

Integrate over M and apply the Green's formula $\int_M u\Delta u dx = \int_M |\nabla u|^2 dx$ we get

$$0 \geq \int_M \left(\frac{\lambda}{m} - \lambda + (m-1)C\right)|\nabla u|^2 dx.$$

This implies

$$\frac{\lambda}{m} - \lambda + (m-1)C \leq 0,$$

i.e.

$$\lambda \geq mC.$$

□

One can prove that the first eigenvalue of the standard sphere S^m is m. In fact, this is the only case where $\lambda_1 = m$ if (M, g) satisfies the conditions in the above theorem.

Theorem 2.3 (Obata). *Let (M, g) be a closed Riemannian manifold with $Ric \geq (m-1)C$ for some $C > 0$. If $\lambda_1 = mC$, then (M, g) is isometric to the round sphere $(S^m(\frac{1}{\sqrt{C}}), g_{round})$.*

Proof. Without loss of generality we may assume $C = 1$. If $\lambda_1 = m$, then from the proof above we see

$$Ric(\nabla u, \nabla u) = (m-1)|\nabla u|^2.$$

Since $\Delta(u^2) = 2u\Delta u - 2|\nabla u|^2$ (see PSet 1), from (1) we get

$$-\frac{1}{2}\Delta\left(|\nabla u|^2 + u^2\right) \geq u\Delta u - m|\nabla u|^2 + (m-1)|\nabla u|^2 - u\Delta u + |\nabla u|^2 = 0.$$

It follows $\Delta\left(|\nabla u|^2 + u^2\right) \equiv 0$ since its integral over M is 0. In other words,

$$|\nabla u|^2 + u^2 = constant.$$

We normalize u so that $\max_M u^2 = 1$. Since $\nabla u = 0$ at the maximum/minimum points of u, we get

$$|\nabla u|^2 + u^2 = 1 \qquad \text{and} \qquad \max_M u = -\min_M = 1.$$

Now let $p, q \in M$ be points such that $u(p) = -1, u(q) = 1$. Let $l = d(p, q)$ and let $\gamma : [0, l] \to M$ be a normal geodesic from p to q. Let $f(t) = u(\gamma(t))$. Then

$$\frac{|f'(t)|}{\sqrt{1 - f^2(t)}} \leq \frac{|\nabla u(\gamma(t))|}{\sqrt{1 - u(\gamma(t))^2}} = 1.$$

Integrating, we get

$$\pi \leq d(p, q).$$

So $diam(M, g) \geq \pi$. But by Bonnet-Meyer, $diam(M, g) \leq \pi$. So $diam(M, g) = \pi$. Finally by Cheng's maximal diameter theorem, (M, g) is isomorphic to the standard sphere. $\square$

Spectral geometry is the branch of differential geometry that studies the relations between the spectrum of the Laplace-type operator and the underline geometry. There are many many beautiful results that have been proved, and at the meantime there are also many many open problems to be studied in the future.