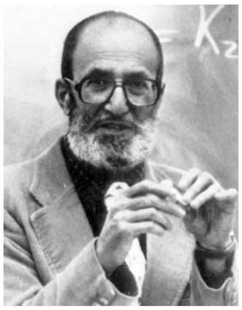
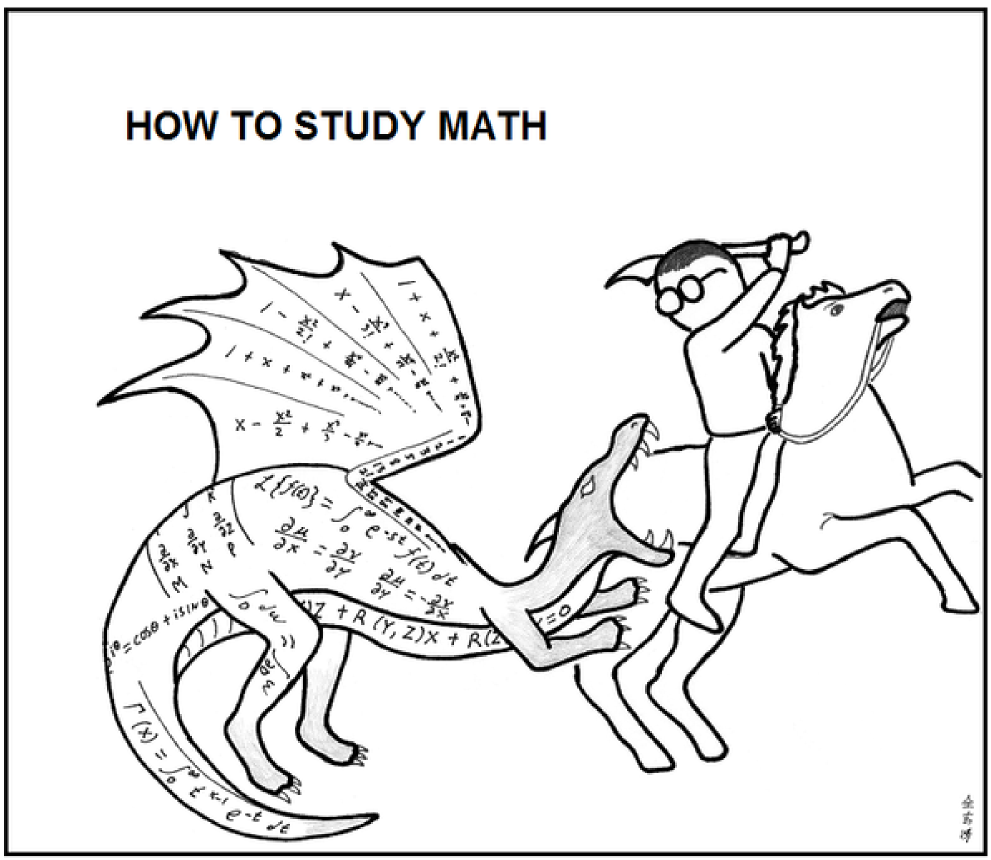


Don't just read it; fight it!
Ask your own questions,
look for your own examples,
discover your own proofs.
Is the hypothesis necessary?
Is the converse true?
What happens in the classical special case?
What about the degenerate cases?
Where does the proof use the hypothesis?



Don't just read it; fight it!

--- Paul R. Halmos

Course Information

♣ About the course:

- **Instructor:** Zuoqin Wang
  - **Email:** wangzuoq@ustc.edu.cn
  - **Office:** 1601
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  - Yiyu Wang, wangyiyu@mail.ustc.edu.cn
  - Yulin Gong, gylustc@mail.ustc.edu.cn
  - **Question Answering:** Sundays 3:55-5:30pm @ 5207
- **Lecture time/room:** T 3:55-5:30 pm & Th 9:45-11:20 am @ 1302
- **Webpage:** http://staff.ustc.edu.cn/∼wangzuoq/Courses/18F-Manifolds/
- **PSets:** Will be posted every week on the course webpage, and will be collected every two weeks (on Thursdays before class).
- **Exams:** There will be one midterm and one final exam.
- **Language:** We will use English in all Lectures, PSets and Exams.
- **Course grades**:
  - **Regular:** HWs (30%), Midterm (30%), Final (40%)
  - **To get A+:** An additional term paper is required. (More details after midterm)

♢ Notes and Reference books:

Course Notes will be uploaded to the course webpage after each lecture.

We will not follow any single book. The following are some nice reference books:

- *Introduction to Smooth Manifolds, 2nd ed,* by John Lee
- *An Introduction to Manifolds, 2nd ed,* by Loring W. Tu
- *Differential Topology*, by Victor Guillemin and Alan Pollack

♡ Prerequisites:

- *Basic Analysis:* C^k maps, multiple integrals, the inverse and implicit function theorems, existence and uniqueness theory for ODEs
  ——c.f. *Appendix C and Appendix D* in Lee's book
- *Basic Topology:* Topological spaces, quotient spaces, connectedness, compactness, Hausdorff, second countable, continuity, proper
  ——c.f. *Appendix A* in Lee's book
- *Basic Algebra:* Linear spaces, direct sums, inner products, linear transformations, matrices, groups, quotient groups
  ——c.f. *Appendix B* in Lee's book

♠ **Contents:**

Smooth manifolds are nice geometric objects on which one can do analysis: they are higher dimensional generalizations of smooth curves and smooth surfaces; they appear as the solution sets of systems of equations, the phase spaces of many physics system, etc. They are among the most important objects in modern mathematics and physics.

In this course we plan to cover

- Basic theory: definitions, examples, structural theorems etc.
  - Smooth manifolds and submanifolds
  - Smooth maps and differentials
  - Vector fields and flows
  - Basic differential topology
  - Manifolds with special structures: Lie groups, vector/fiber bundles
- Geometry of differential forms
  - Tensors, differential forms
  - Exterior derivatives and integrations of differential forms
  - de Rham cohomology and applications
  - Geometric structures related to differential forms: Riemannian/symplectic structures, Chern-Weil(if time permit)

LECTURE 1: TOPOLOGICAL MANIFOLDS

1. Review of Topology

Recall that a *topology* on a set X is a collection $\mathcal{O}$ of subsets of X whose elements are called *open sets*, such that

- The set X and the empty set $\emptyset$ are open sets.
- Any union of open sets is an open set.
- Any finite intersection of open sets is an open set.

As usual, the complement of an open set is called a *closed set*. It is easy to see that if $\mathcal{O}$ is a topology on X, and $Y \subset X$, then

$$\mathcal{O}_Y = \{O \cap Y \mid O \subset \mathcal{O}\}$$

is a topology on Y. This is called the *induced subspace topology*.

We will only study "nice" topological spaces. Recall that a topological space X is

- *Hausdorff* (i.e. T_2) if for any $x \neq y \in X$, there exist open sets $U \ni x$ and $V \ni y$ so that $U \cap V = \emptyset$.
- *second-countable* (i.e. A_2) if there exists a countable sub-collection $\mathcal{O}_0$ of $\mathcal{O}$ so that any open set is a union of (not necessarily finite) open sets in $\mathcal{O}_0$.

Unless otherwise stated, all spaces we are going to study in this course will be Hausdorff and second countable. Note that if X is Hausdorff or second-countable, and $A \subset X$, then A is also Hausdorff or second-countable with respect to the induced subspace topology.

There are two more topological conceptions that we will frequently use in this course: the compactness and connectedness. Recall that a subset D in a topological space X is *compact* if for any collection of open sets $\{U_\alpha\}$ satisfying $D \subset \cup_\alpha U_\alpha$, there exists a finite sub-collection $\{U_{\alpha_1}, \cdots, U_{\alpha_k}\}$ so that $D \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$. Usually the compactness will make our life much easier.

Finally we recall the conception of connectedness. A topological space X is said to be *disconnected* if there exist two non-empty open sets U_1 and U_2 in X so that

$$U_1 \cup U_2 = X \quad \text{and} \quad U_1 \cap U_2 = \emptyset.$$

It is called *connected* if it is not disconnected. (If X is disconnected, then any maximal connected subset of X is called a *connected component* of X.) Also we call X *path-connected* if for any $p, q \in X$, there is a continuous map $f : [0, 1] \to X$ so that $f(0) = p$ and $f(1) = q$. Such a map is called a *path* from p to q.

Topological spaces are spaces on which one can define continuous maps. Recall that a map $f : X \to Y$ between topological spaces is called *continuous* if for any open set V in Y, the pre-image $f^{-1}(V)$ is an open set in X. Two topological spaces X and Y are *homeomorphic*, if there is a continuous map $f : X \to Y$ which is one-to-one and onto, so that f^{-1} is also continuous. Such a map f is called a *homeomorphism*. Of course this gives us an equivalence relation in the category of all topological spaces.

A quantity (or "mathematical objects" like groups) associated to topological spaces that is invariant under homeomorphisms is usually called a topological invariant. The following elementary theorem tells us that the dimension of Euclidian spaces is a topological invariant:

Theorem 1.1 (Invariance of Domain)**.** *If U is an open set in $\mathbb{R}^n$ and V is an open set in $\mathbb{R}^m$, and $f : U \to V$ is a homeomorphism, then $m = n$.*

Proof of Theorem 1.1 for $n = 1$. Since a homeomorphism maps a connected component to a connected component, we can assume without loss of generality that both U and V are connected. Suppose $m > 1$, and $f : U \to V$ is a homeomorphism. We take any point $p \in U$. Then $f : U \setminus \{p\} \to V \setminus \{f(p)\}$ is also a homeomorphism. But this cannot happen, since $U \setminus \{p\}$ is disconnected, while $V \setminus \{f(p)\}$ is connected! □

We will prove the theorem later in this course, using more sophisticated topological invariants: the de Rham cohomology groups.

2. Topological manifolds

Now we are ready to define topological manifolds. Roughly speaking, topological manifolds are nice topological spaces that locally look like $\mathbb{R}^n$. (So one can try to do analysis modelled on Euclidean spaces.)

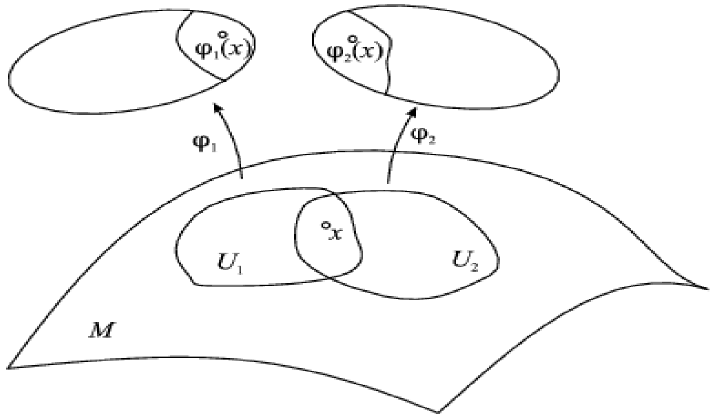
Definition 2.1. An n dimensional *topological manifold* M is a topological space so that

(1) M is Hausdorff.
(2) M is second-countable.
(3) M is locally an Euclidean space of dimension n, i.e. for every $p \in M$, there exists a triple $\{\varphi, U, V\}$, called a *chart* (around p), such that U is an open neighborhood of p in M, V is an open subset of $\mathbb{R}^n$, and $\varphi : U \to V$ is a homeomorphism.

Remark. There are many different charts near any given point p. In fact, suppose (φ, U, V) is a chart near p. Then

(1) for any open neighborhood $U_1 \subset U$ of p, if we denote $\varphi_1 = \varphi|_{U_1}$ and let $V_1 = \varphi(U_1)$, then (φ_1, U_1, V_1) is also a chart near p.

(2) for any $v \in \mathbb{R}^n$, if we let $V_v = V + \{v\}$ and let $\varphi_v(x) = \varphi(x) + v$, then (φ_v, U, V_v) is a chart near p.



Remark. The three conditions in the definition of topological manifold are independent of each other. For example, two crossing lines form a topological space which is Hausdorff and second countable but not locally Euclidean; an uncountable disjoint union of real lines form a Hausdorff and locally Euclidean topological space which is not second countable. In exercise, we will see a topological space which is locally Euclidean (and second countable) but not Hausdorff.

Remark. Both the Hausdorff and the second-countable conditions are important in defining a reasonably nice geometric object. For example, according to the Hausdorff property, the limit of a convergent sequence is unique (try to prove this!). We will prove later that the Hausdorff property together with the second countability property imply the existence of a *partition of unity*, which is a fundamental tool in studying manifolds.

Now we give a couple examples of topological manifolds.

The simplest examples of topological manifolds include the empty set, any set of countable many points, $\mathbb{R}^n$ itself, and open sets in $\mathbb{R}^n$. Here are some more interesting examples of manifolds.

Example. **(Graphs).** For any open set $U \subset \mathbb{R}^m$ and any continuous map $f : U \to \mathbb{R}^n$, the *graph* of f is the subset in $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ defined by

$$\Gamma(f) = \{(x, y) \mid x \in U, y = f(x)\} \subset \mathbb{R}^{m+n}.$$

With the subspace topology inherited from $\mathbb{R}^{m+n}$, $\Gamma(f)$ is Hausdorff and second-countable. It is locally Euclidean since it has a global chart $\{\varphi, \Gamma(f), U\}$, where

$$\varphi : \Gamma(f) \to U, \quad \varphi(x, y) = x$$

is the *projection onto the first factor* map. [To see φ is a homeomorphism: obviously φ is continuous, invertible, and its inverse

$$\varphi^{-1} : U \to \Gamma(f), \quad \varphi^{-1}(x) = (x, f(x))$$

is continuous.] So $\Gamma(f)$ is a topological manifold of dimension m.

Example. **(Spheres).** For each $n \geq 0$, the unit n-sphere

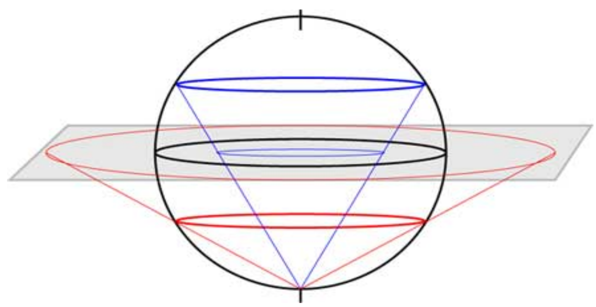
$$S^n = \{(x^1, \cdots, x^n, x^{n+1}) \mid (x^1)^2 + \cdots + (x^n)^2 + (x^{n+1})^2 = 1\} \subset \mathbb{R}^{n+1}$$

with the subspace topology is Hausdorff and second-countable. To show that it is locally Euclidean, we can cover S^n by two open subsets

$$U_+ = S^n \setminus \{(0, \cdots, 0, -1)\}, \qquad U_- = S^n \setminus \{(0, \cdots, 0, 1)\}$$

and define two charts $\{\varphi_+, U_+, \mathbb{R}^n\}$ and $\{\varphi_-, U_-, \mathbb{R}^n\}$ by the *stereographic projections*

$$\varphi_\pm(x^1, \cdots, x^n, x^{n+1}) = \frac{1}{1 \pm x^{n+1}}(x^1, \cdots, x^n).$$



It is easy to check that $\varphi_\pm$ are continuous, invertible, and the inverse

$$\varphi_\pm^{-1}(y^1, \cdots, y^n) = \frac{1}{1 + (y^1)^2 + \cdots + (y^n)^2} \left(2y^1, \cdots, 2y^n, \pm(1 - (y^1)^2 - \cdots - (y^n)^2)\right)$$

is also continuous.

Remark. We can also cover S^n by $2n + 2$ charts using hemispheres. More precisely, for any $1 \leq i \leq n + 1$, we let

$$U_i^+ = \{(x^1, \cdots, x^{n+1}) \in S^n : x^i > 0\}$$

be the "upper hemisphere" in the i^{th} direction and define $\varphi_i^+ : U_i^+ \to B^n(1)$ be the projection map

$$\varphi_i^+(x^1, \cdots, x^{n+1}) = (x^1, \cdots, x^{i-1}, x^{i+1}, x^{n+1}),$$

where $B^n(1)$ is the unit ball in $\mathbb{R}^n$. Then one can check that $(\varphi_i^+, U_i^+, B^n(1))$ are charts. Similarly one can construct charts $(\varphi_i^-, U_i^-, B^n(1))$ on each "lower hemispheres".