LECTURE 2: SMOOTH MANIFOLDS

1. Review of analysis

Let U be an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ a continuous function. Recall that f is said to be a C^k*-function*, if all its partial derivatives of order at most k,

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n \le k$$

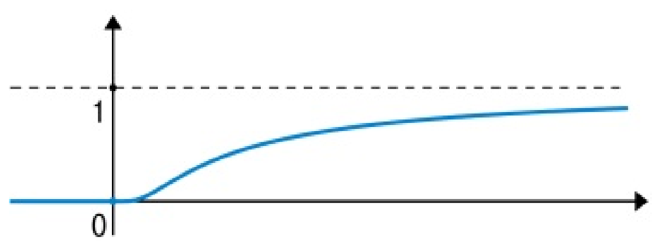
exist and are continuous on U. We say that f is a C^∞ function, or a *smooth function*, if it is of class C^k for all positive integers k. A function f is an *analytic function* (or a C^ω function) if it is smooth and agrees with its Taylor series in a neighborhood of every point. Note that not all smooth functions are analytic.

Example. The function

$$f(x) = \begin{cases} 0, & x \le 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases}$$

is a C^∞ function defined on $\mathbb{R}$, but it is not analytic at $x = 0$. (Check this!)

The graph of this function looks like



Now let U be an open set in $\mathbb{R}^n$ and V be an open set in $\mathbb{R}^m$. Let

$$f = (f_1, \cdots, f_m) : U \to V$$

be a continuous map. We say f is C^∞ (or C^k, or C^ω) if each component f_i, $1 \le i \le m$, is a C^∞ (or C^k, or C^ω) function. [1]

[1]In this course we will mainly consider C^∞ functions/maps. However, most definitions/theorems can be easily extended to the C^k setting. On the other hand, C^ω theory will be quite different.

Let f be a smooth map. The *differential* of f, df, assigns to each point $x \in U$ a linear map $df_x : \mathbb{R}^n \to \mathbb{R}^m$ whose matrix (with respect to the canonical basis) is the *Jacobian matrix* of f at x,

$$df_x = \begin{pmatrix} \frac{\partial f_1}{\partial x^1}(x) & \cdots & \frac{\partial f_1}{\partial x^n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x^1}(x) & \cdots & \frac{\partial f_m}{\partial x^n}(x) \end{pmatrix}.$$

Note that for any $v \in \mathbb{R}^n$, the directional derivative of f at x in the direction v is

$$df_x(v) = \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}.$$

One can regard the differential df_x as a "linearization" of the map f near the point x.

The well-known **chain rule** asserts that if $f : U \to V$ and $g : V \to W$ are smooth maps, so is the composition map $g \circ f : U \to W$, and

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Definition 1.1. A smooth map $f : U \to V$ is a *diffeomorphism* if f is one-to-one and onto, and $f^{-1} : V \to U$ is also smooth.

Obviously

- If $f : U \to V$ is a diffeomorphism, so is f^{-1}.
- If $f : U \to V$ and $g : V \to W$ are diffeomorphisms, so is $g \circ f : U \to W$.

As a consequence, we can prove the following smooth "invariance of domain" theorem:

Theorem 1.2 (Invariance of Domain, smooth version)**.** *If $f : U \to V$ is a diffeomorphism, then for each $x \in U$, the differential df_x is a linear isomorphism. In particular,* $\dim U = \dim V$.

Proof. Applying the chain rule to $f^{-1} \circ f = \mathrm{Id}_U$, and notice that the differential of the identity map $\mathrm{Id}_{\mathrm{U}} : U \to U$ is the identity transformation $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$, we get

$$df^{-1}_{f(x)} \circ df_x = \mathrm{Id}_{\mathbb{R}^n}.$$

The same argument applies to $f \circ f^{-1}$, which yields

$$df_x \circ df^{-1}_{f(x)} = \mathrm{Id}_{\mathbb{R}^m}.$$

By basic linear algebra, we conclude that $m = n$ and that df_x is an isomorphism. □

Conversely, one would like to ask: if the linearization df_x is an linear isomorphism, what can we say about the map f itself? In general f is no longer a diffeomorphism:

Example. Consider the map

$$f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}, \quad (x^1, x^2) \mapsto ((x^1)^2 - (x^2)^2, 2x^1x^2).$$

Then at each point $x \in \mathbb{R}^2 \setminus \{0\}$,

$$df_x = \begin{pmatrix} 2x^1 & -2x^2 \\ 2x^2 & 2x^1 \end{pmatrix}.$$

Since $x = (x^1, x^2) \neq (0,0)$, df_x is an isomorphism for each x. However, f is not invertible since $f(x) = f(-x)$.

To get a better understanding of this example, one can identify $\mathbb{R}^2$ with $\mathbb{C}$. Then f is actually the complex function

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, \quad z \mapsto f(z) = z^2.$$

In particular, we see that although f itself is not a diffeomorphism, it is not very far away from being a diffeomorphism: for any $x \in \mathbb{C} \setminus \{0\}$, one can find a small neighborhood U_x of x, such that the restriction $f|_{U_x} : U_x \to f(U_x)$ is a diffeomorphism. This motivates the following definition:

Definition 1.3. Let $f : U \to V$ be a smooth map. We say f is a *local diffeomorphism* near $x \in U$ if there exists a neighborhood U_x containing x and a neighborhood $V_{f(x)}$ containing $f(x)$ such that

$$f|_{U_x} : U_x \to V_{f(x)}$$

is a diffeomorphism

Now we can state the inverse function theorem, which can be viewed as a partial inverse of the "smooth invariance of domain" theorem.

Theorem 1.4 (The inverse function theorem). *If $f : U \to V$ is a smooth map, and df_x is an isomorphism, then f is a* local diffeomorphism *near x.*

In other words, an isomorphism in the linear category implies a *local* diffeomorphism in the smooth category. The inverse function theorem is a special case (why) of the following implicit function theorem that we learned from multivariable analysis course:

Theorem 1.5 (The implicit function theorem). *Let W be an open set in $\mathbb{R}^n_x \times \mathbb{R}^m_y$, and $F = (F_1, \cdots, F_m) : W \to \mathbb{R}^m$ a smooth map. Let (x_0, y_0) be a point in W so that the $m \times m$ matrix*

$$\begin{pmatrix} \frac{\partial F_1}{\partial y^1}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial y^m}(x_0, y_0) \\ \vdots & \vdots & \vdots \\ \frac{\partial F_m}{\partial y^m}(x_0, y_0) & \cdots & \frac{\partial F_m}{\partial y^m}(x_0, y_0) \end{pmatrix}.$$

is nonsingular, then there exists a neighborhood $U_0 \times V_0$ of (x_0, y_0) in W and a smooth map $f : U_0 \to V_0$ so that the graph of f is the level set of F near (x_0, y_0), i.e.

- *$f(x_0) = y_0$,*
- *If we denote $c = F(x_0, y_0)$, then $F^{-1}(c) \cap (U_0 \times V_0)$ is the graph of f, i.e. $F(x, f(x)) = c$ for all $x \in U_0$.*

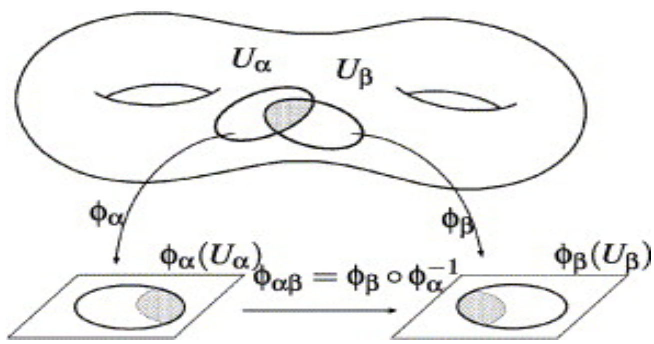
2. Smooth Manifolds

We would like to define smooth structures on topological manifolds so that one can do calculus on it. In particular, we should be able to talk about smoothness of continuous functions on a given smooth manifold M. Since near each point in M, one has a chart $\{\varphi, U, V\}$ which identify the open set U in M with the open set V in $\mathbb{R}^n$, it is natural identify any function f on U with the function $f \circ \varphi^{-1}$ on V, and use the smoothness of $f \circ \varphi^{-1}$ to define the smoothness of f itself. This idea is of course correct. The only issue is that a point on M could sit in many different open charts, and the smoothness of a function at this point should be independent of the choice of charts. In other words, if both φ and ψ are chart maps near a point, we want the maps $f \circ \varphi^{-1}$ and $f \circ \psi^{-1}$ to be simultaneously smooth or non-smooth. This amounts to require $\varphi \circ \psi^{-1}$ to be smooth. With this as our intuition, we define

Definition 2.1. Let M be a topological manifold of dimension n. We say two charts $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ and $\{\varphi_\beta, U_\beta, V_\beta\}$ of M are *compatible* if the *transition map*

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism. (Note that both $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$, so the smoothness of $\varphi_{\alpha\beta}$ is well understood.)



Example. **(The Spheres - continued).** Recall that the n-sphere S^n admits two atlas $\{\varphi_+, U_+, \mathbb{R}^n\}$ and $\{\varphi_-, U_-, \mathbb{R}^n\}$, where

$$U_\pm = S^n \setminus \{(0, \cdots, 0, \mp 1)\}$$

and

$$\varphi_\pm(x^1, \cdots, x^n, x^{n+1}) = \frac{1}{1 \pm x^{n+1}}(x^1, \cdots, x^n).$$

Moreover the inverses of $\varphi_\pm$ are given by

$$\varphi_\pm^{-1}(y^1, \cdots, y^n) = \frac{1}{1+|y|^2}\left(2y^1, \cdots, 2y^n, \pm(1-|y|^2)\right)$$

It follows that on $\varphi_-(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned}\varphi_{-+}(y^1, \cdots, y^n) &= \varphi_+ \circ \varphi_-^{-1}(y^1, \cdots, y^n) \\ &= \varphi_+ \left(\frac{1}{1+|y|^2} \left(2y^1, \cdots, 2y^n, -1+|y|^2\right)\right) \\ &= \frac{1}{|y|^2}(y^1, \cdots, y^n),\end{aligned}$$

which is a diffeomorphism from $\mathbb{R}^n \setminus \{0\}$ to itself. So these two charts are compatible. [Check: charts of S^n defined via hemispheres are compatible with these two charts.]

Definition 2.2. (1) An *atlas* $\mathcal{A}$ on M is a collection of charts $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ with $\bigcup_\alpha U_\alpha = M$, such that all charts in $\mathcal{A}$ are compatible to each other.
(2) Two atlas on M are said to be *equivalent* if their union is still an atlas on M.

Example. We can define two atlas on $\mathbb{R}$ by $\mathcal{A} = \{\varphi_1, \mathbb{R}, \mathbb{R}\}$ and $\mathcal{B} = \{\varphi_2, \mathbb{R}, \mathbb{R}\}$, where $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$. Then they are non-equivalent atlas since

$$\varphi_{21}(x) = \varphi_1 \circ \varphi_2^{-1}(x) = x^{1/3}$$

is not smooth on $\mathbb{R}$.

Definition 2.3. An n-dimensional *smooth manifold* is an n-dimensional topological manifold M equipped with an equivalence class of atlas. This equivalence class is called its *smooth structure*.

So a smooth manifold is a pair $(M, \mathcal{A})$. In the future we will always omit $\mathcal{A}$ if there is no confusion of the smooth structure.

Example. If a topological manifold M can be covered by a single chart, then such a chart automatically determines a smooth structure on M.

Here is an interesting (maybe weird?) consequence: let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be any <u>continuous</u> function. Then the graph $\Gamma(f)$ admits an intrinsic structure of a <u>smooth</u> manifold. (However, it is possible that $\Gamma(f)$ is not a *smooth submanifold* of $\mathbb{R}^{n+1}$.)

Example. **(The Real Projective Spaces).**

The n dimensional *real projective space* $\mathbb{RP}^n$ is by definition the set of lines passing the origin in $\mathbb{R}^{n+1}$. To give $\mathbb{RP}^n$ a topology, we will regard it as the quotient space

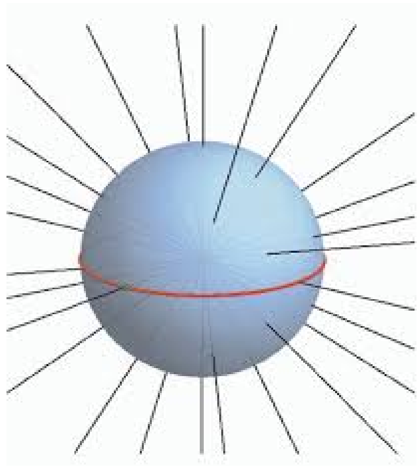
$$\mathbb{RP}^n = \mathbb{R}^{n+1} - \{(0, \cdots, 0)\}/\sim,$$

where the equivalent relation $\sim$ is given by

$$(x^1, \cdots, x^{n+1}) \sim (tx^1, \cdots, tx^{n+1}), \qquad \forall t \neq 0.$$

One can also regard $\mathbb{RP}^n$ as the quotient of S^n by gluing the antipodal points

$$\mathbb{RP}^n = S^n / \sim .$$



Form these descriptions it is easy to see that $\mathbb{RP}^n$ is Hausdorff and second-countable, and in fact is compact.

Usually people will denote the element (=the equivalence class or the "line") in $\mathbb{RP}^n$ containing the point $(x^1, \cdots, x^{n+1})$ by $[x^1 : \cdots : x^{n+1}]$.

Now we construct local charts on $\mathbb{RP}^n$. Consider the open sets

$$U_i = \{[x^1 : \cdots : x^{n+1}] \mid x^i \neq 0\}, \qquad 1 \leq i \leq n+1.$$

For each i, define $\varphi_i : U_i \to \mathbb{R}^n$ to be

$$\varphi_i([x^1 : \cdots : x^{n+1}]) = \left(\frac{x^1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x^{n+1}}{x^i}\right).$$

It is not hard to check that each φ_i is well-defined, is continuous, and the inverse

$$\varphi_i^{-1}(y^1, \cdots, y^n) = [y^1 : \cdots : y^{i-1} : 1 : y^i : \cdots : y^n].$$

is continuous. So each $(\varphi_i, U_i, \mathbb{R}^n)$ is a chart.

Finally we will show that $\mathbb{RP}^n$ is in fact a smooth manifold. Without loss of generality, let's verify that $\varphi_{1,n+1}$ is a diffeomorphism between

$$\varphi_1(U_1 \cap U_{n+1}) = \{(y^1, \cdots, y^n) \mid y^n \neq 0\} =: V_n$$

and

$$\varphi_{n+1}(U_1 \cap U_{n+1}) = \{(y^1, \cdots, y^n) \mid y^1 \neq 0\} =: V_1.$$

In fact, by definition

$$\begin{aligned}\varphi_{1,n+1}(y^1, \cdots, y^n) &= \varphi_{n+1} \circ \varphi_1^{-1}(y^1, \cdots, y^n) \\ &= \varphi_{n+1}([1 : y^1 : \cdots : y^n]) \\ &= \left(\frac{1}{y^n}, \frac{y^1}{y^n}, \cdots, \frac{y^{n-1}}{y^n}\right)\end{aligned}$$

which is obviously a diffeomorphism from V_n to V_1. Similarly one can show that all other transition maps φ_{ij} are diffeomorphisms.

(By a similar way one can define the n dimensional *complex projective space* $\mathbb{CP}^n$ as the space of "complex lines" in $\mathbb{C}^n$ and verify that it is a smooth manifold.)

Remark. Similarly one can define C^k manifolds, real analytic ($=C^\omega$) manifolds and complex manifolds. For example, a complex manifold is a Hausdorff and second countable topological space that locally looks like $\mathbb{C}^n$, so that the transition maps are all bi-holomorphic.

Remark. There exist topological manifolds that do not admit smooth structures. The first example is a compact 10-dimensional manifold found by M. Kervaire.