LECTURE 3: SMOOTH FUNCTIONS

1. Smooth Functions

Definition 1.1. Let $(M, \mathcal{A})$ be a smooth manifold, and $f : M \to \mathbb{R}$ a function.

(1) We say f is *smooth at* $p \in M$ if there exists a chart $\{\varphi_\alpha, U_\alpha, V_\alpha\} \in \mathcal{A}$ with $p \in U_\alpha$, such that the function $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is smooth at $\varphi_\alpha(p)$.
(2) We say f is a *smooth function* on M if it is smooth at every $x \in M$.

Remark. Suppose $f \circ \varphi_\alpha^{-1}$ is smooth at $\varphi(p)$. Let $\{\varphi_\beta, U_\beta, V_\beta\}$ be another chart in $\mathcal{A}$ with $p \in U_\beta$. Then by the compatibility of charts, the function

$$f \circ \varphi_\beta^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1})$$

must be smooth at $\varphi_\beta(p)$. So the smoothness of a function is independent of the choice of charts in the given atlas.

Remark. According to the chain rule, it is easy to see that if $f : M \to \mathbb{R}$ is smooth at $p \in M$, and $h : \mathbb{R} \to \mathbb{R}$ is smooth at $f(p)$, then $h \circ f$ is smooth at p.

We will denote the set of all smooth functions on M by $C^\infty(M)$. Note that this is a (commutative) algebra, i.e. it is a vector space equipped with a (commutative) bilinear "multiplication operation": If f, g are smooth, so are $af + bg$ and fg; moreover, the multiplication is commutative, associative and satisfies the usual distributive laws.

Example. Each coordinate function $f_i(x^1, \cdots, x^{n+1}) = x^i$ is a smooth function on S^n, since

$$f_i \circ \varphi_\pm^{-1}(y^1, \cdots, y^n) = \begin{cases} \frac{2y^i}{1+|y|^2}, & 1 \le i \le n \\ \pm\frac{1-|y|^2}{1+|y|^2}, & i = n+1 \end{cases}$$

are smooth functions on $\mathbb{R}^n$.

Now suppose $f \in C^\infty(M)$. As usual, the *support* of f is by definition the set

$$\mathrm{supp}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}.$$

We say that f is *compactly supported*, denoted by $f \in C_0^\infty(M)$, if the support of f is a compact subset in M. Obviously

- If $f, g \in C_0^\infty(M)$, then $af + bg \in C_0^\infty(M)$.
- If $f \in C_0^\infty(M)$ and $g \in C^\infty(M)$, then $fg \in C_0^\infty(M)$.

So $C_0^\infty(M)$ is an ideal of of the algebra $C^\infty(M)$. Note that if M is compact, then any smooth function is compactly supported.

Example (Bump function). A *bump function* (sometimes also called a *test function*) is a compactly supported smooth function, which is usually supposed to be non-negative, no more than 1, and equals to 1 on a given compact set.

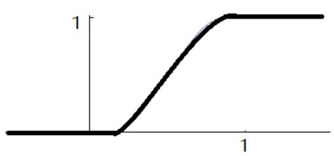
Here is a how we construct a bump function on $\mathbb{R}^n$: We will first define two auxiliary functions f_1 and f_2 on $\mathbb{R}$. Then we define the bump function f_3 on $\mathbb{R}^n$. In what follows we list the definition of f_k in the left, and list the properties of f_k in the right. The smoothness and properties of f_k follows from that of f_{k-1}:

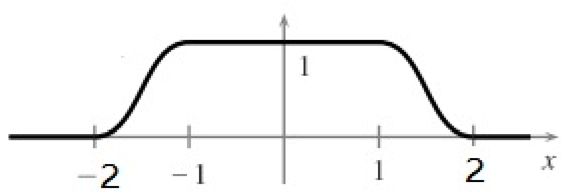
$$f_1(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0 \end{cases} \qquad \Longrightarrow f_1(x) = \begin{cases} \in (0,1), & x > 0, \\ 0, & x \le 0, \end{cases}$$

$$f_2(x) = \frac{f_1(x)}{f_1(x) + f_1(1-x)} \qquad \Longrightarrow f_2(x) = \begin{cases} 0, & x \le 0, \\ \in (0,1), & 0 < x < 1, \\ 1, & x \ge 1 \end{cases}$$

$$f_3(x) = f_2(2 - |x|) \qquad \Longrightarrow f_3(x) = \begin{cases} 0, & |x| \ge 2, \\ \in (0,1), & 1 < |x| < 2, \\ 1, & |x| \le 1. \end{cases}$$

We have seen the graph of f_1. Here are the graphs of f_2 and f_3 (with $n = 1$):





With the help of the Euclidean bump functions, we can construct bump functions on any smooth manifold with prescribed support and prescribed "equal to one region":

Theorem 1.2. *Let M be a smooth manifold, $A \subset M$ is a compact subset, and $U \subset M$ an open subset that contains A. Then there is a bump function $\varphi \in C_0^\infty(M)$ so that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on A and* $\mathrm{supp}(\varphi) \subset U$.

Proof. [The idea of the proof: Cover the compact set A by finitely many small pieces, where each piece is contained in one (carefully chosen) chart, so that one can copy the "Euclidean bump function" that we constructed above to such pieces.]

For each $q \in A$, there is a chart $\{\varphi_q, U_q, V_q\}$ near q so that $U_q \subset U$ and V_q contains the open ball $B_3(0)$ of radius 3 centered at 0 in $\mathbb{R}^n$.[1] Let $\widetilde{U}_q = \varphi_q^{-1}(B_1(0))$, and let

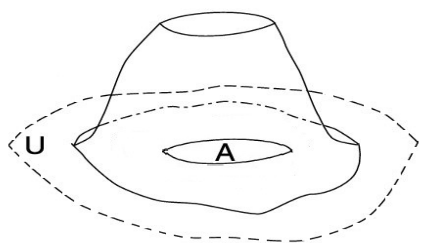
$$f_q(p) = \begin{cases} f_3(\varphi_q(p)), & p \in U_q, \\ 0, & p \notin U_q. \end{cases}$$

[1]Think about this: Why one can find such a chart which is compatible with given charts?

Then $f_q \in C_0^\infty(M)$, $\mathrm{supp}(f_q) \subset U_p$ and $f \equiv 1$ on $\widetilde{U}_q$. (Which assumption do we need here?)

Now the family of open sets $\{\widetilde{U}_q\}_{q\in A}$ is an open cover of A. Since A is compact, there is a finite sub-cover $\{\widetilde{U}_{q_1}, \cdots, \widetilde{U}_{q_N}\}$. Let $\psi = \sum_{i=1}^N f_{q_i}$. Then ψ is a smooth and compactly supported function on M so that $\psi \geq 1$ on A and $\mathrm{supp}(\psi) \subset U$. It follows that the function $\varphi(p) = f_2(\psi(p))$ satisfies all the conditions we want. □

Here is what such a bump function will look like:



As a simple consequence, we see that the vector space $C_0^\infty(M)$ (and thus $C^\infty(M)$) is infinitely dimensional (assuming $\dim M > 0$).

2. Partition of unity

As we have just seen, for a compact subset $K \subset M$, one can always cover it by finitely many nice neighborhoods on which we can construct nice "local" functions. By adding these (finitely many) local functions, we can get nice global functions on M that behaves nicely on K. It turns out that the same idea applies to the whole manifold M: we can generate an infinite collection of smooth functions on M, and add them to get a global smooth function, provide that near each point, there are only finitely many functions in our collection that are nonzero. More importantly, we can use such a collection of functions to "glue" geometric/analytic objects that can be defined locally using charts.

Definition 2.1. Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. A *partition of unity* (P.O.U. in brief) *subordinate to the cover* $\{U_\alpha\}$ is a collection of smooth functions $\{\rho_\alpha\}$ so that

(1) $0 \leq \rho_\alpha \leq 1$ for all α.
(2) $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$ for all α.
(3) each point $p \in M$ has a neighborhood which intersects $\mathrm{supp}(\rho_\alpha)$ for only finitely many α.
(4) $\sum_\alpha \rho_\alpha(p) = 1$ for all $p \in M$.

Remark. Two consequences of the local finiteness condition (3): Let denote by W_p a neighborhood of p which intersect only finitely many $\mathrm{supp}(\rho_\alpha)$'s.

- Since $\{W_p\}_{p\in M}$ is an open cover of M, and since M is second countable, one can find countably many W_{p_i}'s which also cover M. Since each W_{p_i} intersect only finitely many $\text{supp}(\rho_\alpha)$'s, we conclude that there are only countable many ρ_α's whose support are non-empty. So even if we may start with uncountably many open sets, the P.O.U. automatically "delete" most of them so that only countably many of them are left (which still form an open cover of M).
- For each p, on the open set W_p the sum in (4) [which looks like an uncountable sum, or maybe a countable infinite sum in view of the previous paragraph] is in fact a finite sum.

P.O.U. is one of the main tools in this course. Next time we will prove the following fundamental theorem:

Theorem 2.2 (Existence of P.O.U.). *Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. Then there exists a P.O.U. subordinate to $\{U_\alpha\}$.*

Locally each manifold looks like $\mathbb{R}^n$, so that one have rich mathematics on it. P.O.U. is the tool that can "glue" local smooth objects in a global smooth object. We will see many such examples in the future. For example, we will define integrals of differential forms in local charts, and use P.O.U. to define the integral of a differential form on the whole manifold.

To apply P.O.U., one usually need to carefully choose an open cover. To illustrate this, we give a couple examples below.

As a first consequence of P.O.U., we generalize Theorem 1.2 to closed subsets:

Corollary 2.3. *Let M be a smooth manifold, $A \subset M$ is a closed subset, and $U \subset M$ an open subset that contains A. Then there is a "bump" function $\varphi \in C^\infty(M)$ so that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on A and $\text{supp}(\varphi) \subset U$.*

Proof. Obviously $\{U, M\backslash A\}$ is an open cover of M. Let $\{\rho_1, \rho_2\}$ be a P.O.U. subordinate to this open cover. Then the function $\varphi = \rho_1$ is what we need: ρ_1 is smooth, $0 \le \rho_1 \le 1$, $\text{supp}(\rho_1) \subset U$, and finally $\rho_1 = 1$ on A since $\rho_2 = 0$ on A. □

As another application of P.O.U., next time we will prove the following

Theorem 2.4 (Whitney Approximation Theorem). *Let M be a smooth manifold. Then for any continuous function $g : M \to \mathbb{R}$ and any positive continuous function $\delta : M \to \mathbb{R}_{>0}$, there exists a smooth function $f : M \to \mathbb{R}$ so that $|f(p) - g(p)| < \delta(p)$ holds for all $p \in M$.*

[Here is the idea of the proof: For each p one can find a tiny small open set U_p containing p so that g is "almost constant" on U_p. Then on U_p one can approximate g by the constant function $f(q) = g(p)$ (for any $q \in U_p$). Then "glue" all these constant functions together via a P.O.U. ρ_p subordinate to the open cover $\{U_p\}$.]