LECTURE 4: PARTITION OF UNITY

1. Existence of P.O.U.

We will prove the following fundamental theorem:

Theorem 1.1 (Existence of P.O.U.)**.** *Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. Then there exists a P.O.U. subordinate to $\{U_\alpha\}$.*

In other words, we need to find a family $\{\rho_\alpha\}$ of smooth functions so that

(1) $0 \le \rho_\alpha \le 1$ for all α.
(2) $\text{supp}(\rho_\alpha) \subset U_\alpha$ for all α.
(3) each point $p \in M$ has a neighborhood which intersects $\text{supp}(\rho_\alpha)$ for only finitely many α.
(4) $\sum_\alpha \rho_\alpha(p) = 1$ for all $p \in M$.

The proof depends on the following technical lemma.

Lemma 1.2. *Let M be any topological manifold. For any open cover $\mathcal{U} = \{U_\alpha\}$ of M, one can find two countable family of open covers $\mathcal{V} = \{V_j\}$ and $\mathcal{W} = \{W_j\}$ of M so that*

- *For each j, $\overline{V}_j$ is compact and $\overline{V}_j \subset W_j$.*
- *$\mathcal{W}$ is a* refinement *of $\mathcal{U}$: For each j, there is an $\alpha = \alpha(j)$ so that $W_j \subset U_\alpha$.*
- *$\mathcal{W}$ is a* locally finite *cover: Any $p \in M$ has a neighborhood W such that $W \cap W_j \neq \varnothing$ for only finitely many W_j's.*

Remark. A topological space X is called *paracompact* if every open cover admits a locally finite open refinement.

We will first prove Theorem 1.1, then prove Lemma 1.2.

Proof of Theorem 1.1. [Please compare the first paragraph of this proof with the proof of Theorem 1.2 in Lecture 3.] Since $\overline{V}_j$ is compact and $\overline{V}_j \subset W_j$, according to Theorem 1.2 in Lecture 3 we can find nonnegative functions $\varphi_j \in C_0^\infty(M)$ so that $0 \le \varphi_j \le 1$, $\varphi_j \equiv 1$ on $\overline{V}_j$ and $\text{supp}(\varphi_j) \subset W_j$. Since $\mathcal{W}$ is a locally finite cover, the function

$$\varphi = \sum_j \varphi_j$$

is a well-defined smooth function on M. Since each φ_j is nonnegative, and $\mathcal{V}$ is a cover of M, φ is strictly positive on M. It follows that the functions

$$\psi_j = \frac{\varphi_j}{\varphi}$$

are smooth and satisfy $0 \le \psi_j \le 1$ and $\sum_j \psi_j = 1$.

Next let's re-index the family $\{\psi_j\}$ to get the demanded P.O.U. For each j, we fix an index $\alpha(j)$ so that $W_j \subset U_{\alpha(j)}$, and define

$$\rho_\alpha = \sum_{\alpha(j)=\alpha} \psi_j.$$

Note that the right hand side is a finite sum near each point, so it does define a smooth function. According to Problem Set 1 part 2 problem 4,

$$\mathrm{supp}\rho_\alpha = \overline{\bigcup_{\alpha(j)=\alpha} \mathrm{supp}\psi_j} = \bigcup_{\alpha(j)=\alpha} \overline{\mathrm{supp}\psi_j} \subset U_\alpha.$$

Clearly the family $\{\rho_\alpha\}$ is a P.O.U. subordinate to $\{U_\alpha\}$. □

It remains to prove Lemma 1.2. In particular, we want to prove the existence of locally finite open refinement. The proof is quite geometric:

First we prove

Lemma 1.3. *For any topological manifold M, there exists a countable collection of open sets $\{X_i\}$ so that*

1. *For each j, the closure $\overline{X}_j$ is compact.*
2. *For each j, $\overline{X}_j \subset X_{j+1}$.*
3. *$M = \cup_j X_j$.*

Proof. Since M is second countable, there is a countable basis of the topology of M. Out of this countable collection of open sets, we pick those that have compact closures, and denote them by $Y_1, Y_2, \cdots$. Since M is locally Euclidean, it is easy to see that $\mathcal{Y} = \{Y_j\}$ is an open cover of M.

We let $X_1 = Y_1$. Since $\mathcal{Y}$ is an open cover of $\overline{X}_1$ which is compact, there exist finitely many open sets $Y_{i_1}, \cdots, Y_{i_k}$ so that

$$\overline{X}_1 \subset Y_{i_1} \cup \cdots \cup Y_{i_k}.$$

Let

$$X_2 = Y_2 \cup Y_{i_1} \cup \cdots \cup Y_{i_k}.$$

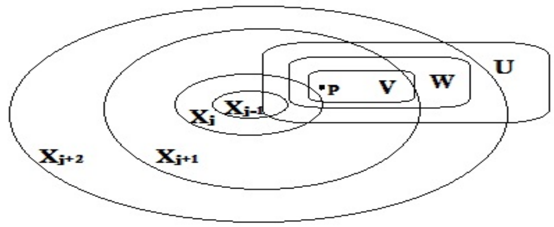
Obviously $\overline{X}_2$ is compact. Repeat this procedure again and again, we could get a sequence of open sets $X_1, X_2, X_3, \cdots$. Obviously the sequence satisfies (1) and (2). It satisfies (3) since $X_k \supset \cup_{j=1}^k Y_j$ □

Remark. Such a collection of subsets is called an *exhaustion* of M.

Proof of Lemma 1.2. For each $p \in M$, there is an j and an $\alpha(p)$ so that $p \in \overline{X}_{j+1} \setminus X_j$ and $p \in U_{\alpha(p)}$. Since M is locally Euclidean, one can always choose open neighborhoods V_p, W_p of p so that $\overline{V}_p$ is compact and

$$\overline{V}_p \subset W_p \subset U_{\alpha(p)} \cap (X_{j+2} \setminus \overline{X}_{j-1}).$$

Now for each j, since the "stripe" $\overline{X}_{j+1} \setminus X_j$ is compact, one can choose finitely many points $p_1^j, \cdots, p_{k_j}^j$ so that $V_{p_1^j}, \cdots, V_{p_{k_j}^j}$ is an open cover of $\overline{X}_{j+1} \setminus X_j$. Denote all



these $V_{p^j_k}$'s by $V_1, V_2, \cdots$, and the corresponding $W_{p^j_k}$'s by $W_1, W_2, \cdots$. Then $\mathcal{V} = \{V_k\}$ and $\mathcal{W} = \{W_k\}$ are open covers of M that satisfies all the conditions in Lemma 1.2. For example, the local finiteness property of $\mathcal{W}$ follows from the fact that there are only finitely many W_k's (that correspond to j and $j-1$ above) intersect $X_{j+1} \setminus \overline{X}_{j-1}$. $\square$

We end with two questions:

- Where did we use the second countable condition in proving P.O.U.?
- Where did we use the Hausdorff condition in proving P.O.U.?

2. An application: Whitney Approximation Theorem

As another application of P.O.U., we prove the following

Theorem 2.1 (Whitney Approximation Theorem). *Let M be a smooth manifold. Then for any continuous function $g : M \to \mathbb{R}$ and any positive continuous function $\delta : M \to \mathbb{R}_{>0}$, there exists a smooth function $f : M \to \mathbb{R}$ so that $|f(p) - g(p)| < \delta(p)$ holds for all $p \in M$.*

In fact we will prove a stronger version of this theorem. Let $A \subset M$ be any closed set. We say a function $g : M \to \mathbb{R}$ is *smooth on A* if there exists an open set $U \supset A$ and a smooth function g_0 defined on U so that $g_0 = g$ on A. [As a consequence, any function g is smooth <u>on</u> any single point set $\{p\}$, although it may not be smooth <u>at</u> p.]

Theorem 2.2 (Whitney Approximation Theorem). *Let M be a smooth manifold, and $A \subset M$ a closed subset. Then for any continuous function $g : M \to \mathbb{R}$ which is smooth on A and any positive continuous function $\delta : M \to \mathbb{R}_{>0}$, there exists $f \in C^\infty(M)$ so that*

$$f(p) = g(p), \qquad \forall p \in A$$

and

$$|f(p) - g(p)| < \delta(p), \qquad \forall p \in M.$$

By taking $A = \emptyset$ we see that Theorem 2.2 implies Theorem 2.1

Proof. [The idea: approximate g by g_0 near A, and approximate g by constant functions elsewhere.] By definition, there exists an open set $U \supset A$ and a smooth function g_0 defined on U so that $g_0 = g$ on A. Let

$$U_0 = \{p \in U : |g_0(p) - g(p)| < \delta(p)\}.$$

Then U_0 is open in M and $U_0 \supset A$.

Next we construct a (nice) open cover of $M\setminus A$. For any $q \in M\setminus A$, we let

$$U_q = \{p \in M\setminus A \ : \ |g(p) - g(q)| < \delta(p)\}.$$

Then $\{U_q \mid q \in M\setminus A\}$ is an open covering of $M\setminus A$.

Now let $\{\rho_0, \rho_q : q \in M\}$ be P.O.U. subordinate to the open cover $\{U_0, U_q : q \in M\}$ of M, and define a function on M via

$$f(p) = \rho_0(p)g_0(p) + \sum_{q\in M} \rho_q(p)g(q).$$

Since the summation is locally finite, f is smooth. Also by definition, $f = g_0 = g$ on A. Moreover, for any $q \in M$ one has

$$\begin{aligned} |f(p) - g(p)| &= \left|\rho_0(p)g_0(p) + \sum_q \rho_q(p)g(q) - \rho_0(p)g(p) - \sum_q \rho_q(p)g(p)\right| \\ &\le \rho_0(p)|g_0(p) - g(p)| + \sum_q \rho_q(p)|g(q) - g(p)| \\ &< \rho_0(p)\delta(p) + \sum_q \rho_q(p)\delta(p) \\ &= \delta(p), \end{aligned}$$

where in the last inequality, the fact $\rho_0(p)|g_0(p) - g(p)| < \rho_0(p)\delta(p)$ follows from the facts that if $p \in U_0$, then by definition $|g_0(p) - g(p)| < \delta(p)$, while if $p \notin U_0$, then $\rho_0(p)$=0; the fact $\rho_q(p)|g(q) - g(p)| < \rho_q(p)\delta(p)$ follows from a similar argument. $\square$