LECTURE 6: LOCAL PROPERTIES OF SMOOTH MAPS

1. Submersions and Immersions

Last time we showed that if $f : M \to N$ is a diffeomorphism, then $df_p : T_pM \to T_{f(p)}N$ is a linear isomorphism. As in the Euclidean case (see Lecture 2), we can prove the following partial converse:

Theorem 1.1 (Inverse Mapping Theorem)**.** *Let $f : M \to N$ be a smooth map such that $df_p : T_pM \to T_{f(p)}N$ is a linear isomorphism, then f is a* local diffeomorphism *near p, i.e. it maps a neighborhood U_1 of p diffeomorphically to a neighborhood X_1 of $q = f(p)$.*

Proof. Take a chart $\{\varphi, U, V\}$ near p and a chart $\{\psi, X, Y\}$ near $f(p)$ so that $f(U) \subset X$ (This is always possible by shrinking U and V). Since $\varphi : U \to V$ and $\psi : X \to Y$ are diffeomorphisms,

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = d\psi_q \circ df_p \circ d\varphi^{-1}_{\varphi(p)} : T_{\varphi(p)}V = \mathbb{R}^n \to T_{\psi(q)}Y = \mathbb{R}^n$$

is a linear isomorphism. It follows from the inverse function theorem in Lecture 2 that there exist neighborhoods V_1 of $\varphi(p)$ and Y_1 of $\psi(q)$ so that $\psi \circ f \circ \varphi^{-1}$ is a diffeomorphism from V_1 to Y_1. Take $U_1 = \varphi^{-1}(V_1)$ and $X_1 = \psi^{-1}(Y_1)$. Then

$$f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$$

is a diffeomorphism from U_1 to X_1. □

Again we cannot conclude global diffeomorphism even if df_p is a linear isomorphism everywhere, since f might not be invertible. In fact, now we can construct an example which is much simpler than the example we constructed in Lecture 2: Let $f : S^1 \to S^1$ be given by $f(e^{i\theta}) = e^{2i\theta}$. Then it is only a local diffeomorphism.

It is natural to ask: what if df_p is not a linear isomorphism? Of course the simplest cases are the full-rank ones.

Definition 1.2. Let $f : M \to N$ be a smooth map.

(1) f is a *submersion* at p if $df_p : T_pM \to T_{f(p)}N$ is surjective.
(2) f is an *immersion* at p if $df_p : T_pM \to T_{f(p)}N$ is injective.

We say f is a submersion/immersion if it is a submersion/immersion at each point.

Obviously

- If f is a submersion, then $\dim M \geq \dim N$.
- If f is an immersion, then $\dim M \leq \dim N$.
- If f is a submersion/immersion at p, then it is a submersion/immersion near p.

Example. If $m \geq n$, then the projection map

$$\pi : \mathbb{R}^m \to \mathbb{R}^n, \quad (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^n)$$

is a submersion

Example. If $m \leq n$, then the inclusion map

$$\iota : \mathbb{R}^m \to \mathbb{R}^n, \quad (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^m, 0, \cdots, 0)$$

is an immersion.

It turns out that any submersion/immersion locally looks like the above two maps.

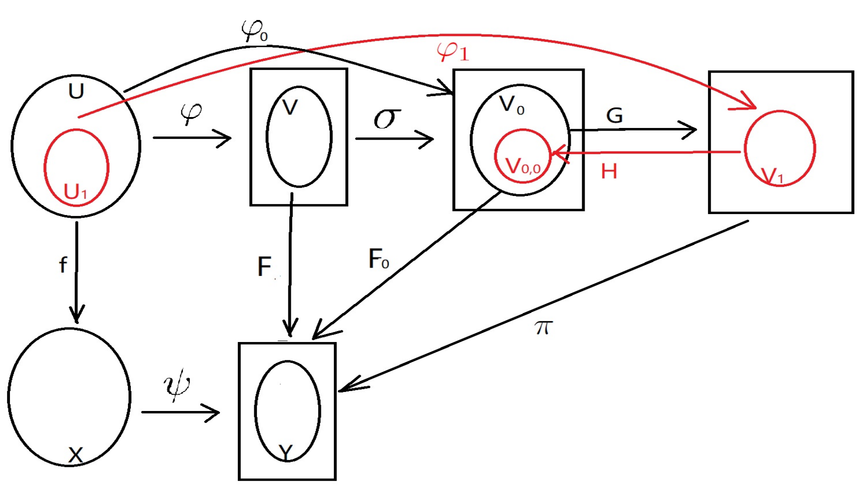
Theorem 1.3 (Canonical Submersion Theorem)**.** *Let $f : M \to N$ be a submersion at $p \in M$, then $m = \dim M \geq n = \dim N$, and there exist charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) around $q = f(p)$ such that*

$$\psi_1 \circ f \circ \varphi_1^{-1} = \pi|_{V_1}.$$

Theorem 1.4 (Canonical Immersion Theorem)**.** *Let $f : M \to N$ be an immersion at $p \in M$, then $m = \dim M \leq n = \dim N$, and there exist charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) around $q = f(p)$ such that*

$$\psi_1 \circ f \circ \varphi_1^{-1} = \iota|_{V_1}.$$

Proof of the Canonical Submersion Theorem. [The construction is shown in the following graph:]



Take a chart $\{\varphi, U, V\}$ near p and a chart $\{\psi, X, Y\}$ near $f(p)$ so that $f(U) \subset X$. Since f is a submersion,

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = d\psi_q \circ df_p \circ d\varphi^{-1}_{\varphi(p)} : T_{\varphi(p)}V = \mathbb{R}^m \to T_{\psi(q)}Y = \mathbb{R}^n$$

is surjective. Denote $F = \psi \circ f \circ \varphi^{-1}$. Then the Jacobian matrix $(\frac{\partial F_i}{\partial x^j})$ is an $n \times m$ matrix of rank n at $\varphi(p)$. By changing coordinates on V (which can be realized by a diffeomorphism $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ that exchange some of the coordinates), we can get a new map $F_0 = F \circ \sigma^{-1} : V_0 = \sigma(V) \to Y$ such that

$$\left(\frac{\partial (F_0)_i}{\partial x^j}\right)_{1\le i,j\le n}$$

is nonsingular at $\sigma(\varphi(p))$. We denote $\varphi_0 = \sigma\circ\varphi$ and $V_0 = \varphi_0(U)$, so that $F_0 = \psi\circ f\circ\varphi_0^{-1}$. Define

$$G : V_0 \to \mathbb{R}^m, \quad (x^1, \cdots, x^m) \mapsto (F_0(x^1, \cdots, x^m), x^{n+1}, \cdots, x^m).$$

Then G satisfies the following two properties (they are the reasons that we define G by the above formula)

(a) $F_0 = \pi \circ G$,

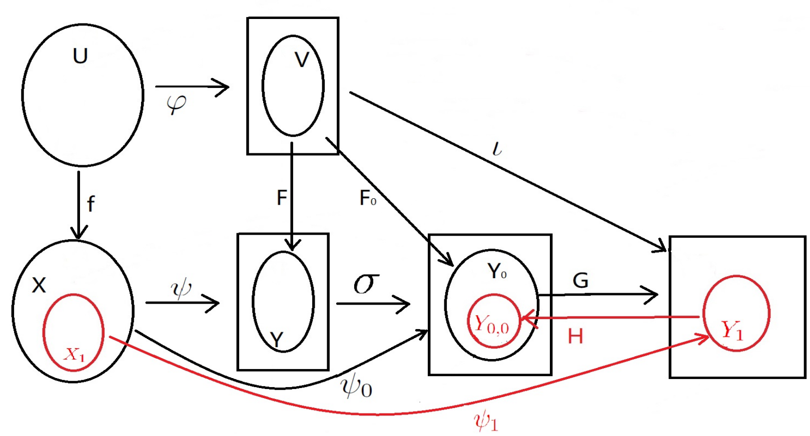
(b) The differential $dG_{\varphi(p)} = \begin{pmatrix} \left(\frac{\partial (F_0)_i}{\partial x^j}\right)_{1\le i,j\le n} & * \\ 0 & \mathrm{Id}_{m-n} \end{pmatrix}$ is nonsingular.

By the inverse function theorem, there is a neighborhood $V_{0,0}$ of $\varphi_0(p)$ in V_0 so that G is a diffeomorphism from $V_{0,0}$ to $V_1 := G(V_{0,0})$. Let H be the inverse of G that maps V_1 to $V_{0,0}$. Let $U_1 = \varphi_0^{-1}(V_{0,0})$ and $\varphi_1 = G \circ \varphi_0$. Then (φ_1, U_1, V_1) is a chart near p, and on V_1 one has

$$\psi \circ f \circ \varphi_1^{-1} = \psi \circ f \circ (\varphi_0^{-1} \circ H) = F_0 \circ H = \pi \circ G \circ H = \pi.$$

This completes the proof. □

Proof of the Canonical Immersion Theorem. The proof is "contained" in the following graph:



where the map H is defined so that

$$\text{(a)} H \circ \iota = F_0, \qquad \text{(b) } dH \text{ is non-singular at } \iota(\varphi(p)),$$

which can be chosen to be

$$H : V \times \mathbb{R}^{n-m} \to \mathbb{R}^n, \quad (x^1, \cdots, x^m, y^1, \cdots, y^{n-m}) \mapsto F(x) + (0, \cdots, 0, y_1, \cdots, y_{n-m}).$$

We will leave the details as an exercise. □

Remark. By using a similar method, one can prove

> **Theorem 1.5** (Constant Rank Theorem). *Let $f : M \to N$ be a smooth map so that* $\text{rank}(df) \equiv r$ *near p. Then there exists charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) near $f(p)$ such that that*
> $$\psi_1 \circ f \circ \varphi_1^{-1}(x^1, \cdots x^m) = (x^1, \cdots, x^r, 0, \cdots, 0).$$

The proof will be left as an exercise.

2. Sard's Theorem

Recall that in calculus, a point a is called a critical point of a smooth function f if all partial derivatives of f at a is zero. More generally, if $f : U \to V$ is a smooth map, we say $a \in U$ is a critical point of f if df_a is not surjective, i.e. f is not a submersion at a. This conception can be easily extended to smooth maps between smooth manifolds:

Definition 2.1. Let $f : M \to N$ be a smooth map.

(1) We say $p \in M$ is a *critical point* of f if df_p is not surjective[1]. We say $p \in M$ is a *regular point* of f if it is not a critical point.
(2) We say $q \in N$ is a *regular value* of f if any $p \in f^{-1}(q)$ is a regular point. We say $q \in N$ is a *critical value* of f if it is not a regular value.

Remark. By definition, any $q \in N \backslash \text{Im}(f)$ is automatically a regular value.

Remark. Critical values are exactly the image of critical points, but the pre-image of critical values may contain regular points. We will denote the set of all critical points of f by $\text{Crit}(f)$.

Example. Let $f : S^n \to \mathbb{R}$ be the "height function",
$$f(x^1, \cdots, x^{n+1}) = x^{n+1}.$$
Then the north pole $(0, \cdots, 0, 1)$ and the south pole $(0, \cdots, 0, -1)$ are critical points of f, and all other points in S^n are regular points of f. The critical values of f are 1 and -1, and all other values in $\mathbb{R}$ are regular values of f.

The following theorem due to Sard is a remarkable theorem in differential topology. The theorem claims that almost all points in the target manifold N are regular values. We will see in the future how to use regular values to construct smooth submanifolds.

Theorem 2.2 (Sard's Theorem). *For any smooth map $f : M \to N$, the set of all critical values is of measure zero in N.*

In particular, if $\dim M < \dim N$, then $f(M)$ is of measure zero in N.

Remark. The theorem does not claim that the set of critical points in M is a measure zero subset. In fact, if we consider a constant map $f(p) \equiv q_0 \in N$, then any point in M is a critical point. However, the set of critical values contains only one point in this case, which is of course of measure zero.

[1]We define critical points to be those points such that $\text{rank}(df_p) < \dim N$. In some books critical points are defined to be those points such that $\text{rank}(df_p) < \min(\dim M, \dim N)$.

We will prove Sard's theorem next time. In the rest of today's lecture, we will explain the words "of measure zero" in the theorem. Note that we have not introduce any measure on M or N yet. One may want to "transplant" the Lebesgue measure on the Euclidean space to manifolds by using local charts. However, this does not give us a well-defined measure on manifolds since it depends on the choice of local charts. In fact, a measure structure is an extra structure on manifold. With only a smooth structure at hand, we don't have a canonical choice of measure structure. (We will endow a smooth manifold with a measure using a volume form in the future).

However, "whether a set is of measure zero or not" make sense without introducing a measure: again the idea is to use the Lebesgue measure on Euclidian space. Recall that a subset $A \subset \mathbb{R}^n$ is of *measure zero* if for any $\varepsilon > 0$, there exists a countable union of open boxes $U_i \in \mathbb{R}^n$ so that

$$A \subset \bigcup_i U_i \qquad \text{and} \qquad \sum_i \text{volume}(U_i) < \varepsilon.$$

For measure zero sets, we have the following properties:

(i) A countable union of measure zero sets is a measure zero set.
(ii) If $A \subset \mathbb{R}^n$ is a measure zero set, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, then $f(A)$ is a measure zero set in $\mathbb{R}^n$. [2]
(iii) Fubini's Theorem: Let A be a countable union of compact sets in $\mathbb{R}^n$ such that the "slice" $A \cap (\{c\} \times \mathbb{R}^{n-r})$ has measure zero in $\mathbb{R}^{n-r}$ for all $c \in \mathbb{R}^r$. Then A has measure zero in $\mathbb{R}^n$.

Since any manifold M can by covered by countable many charts, and each chart identify an open set in M with an open set in $\mathbb{R}^n$, the following definition is independent of coordinate charts (and thus is reasonable):

Definition 2.3. We say $A \subset M$ is a *measure zero* set if for any $p \in A$, one can find a chart (φ, U, V) of M near p so that $\varphi(A \cap U)$ is a measure zero set in V.

[2] In real analysis we know that a continuous function could map a measure zero set in $\mathbb{R}^n$ to a set with positive measure in $\mathbb{R}^n$. However, a (local) Lipschitz map will always map a measure zero set in $\mathbb{R}^n$ to a measure zero set in $\mathbb{R}^n$.