LECTURE 8: SMOOTH SUBMANIFOLDS

1. Smooth submanifolds

Let M be a smooth manifold of dimension n. What object can be called a "smooth submanifold" of M? (Recall: what is a vector subspace W of a vector space V? W should satisfy three conditions: W is a subset of V; W is a vector space by itself; the vector space structure on W should be the restriction of the vector space structure on V. Similarly, what is a subgroup of a group? What is a topological subspace of a topological space? In each case you can always write down the three conditions: the inclusion relation, the structure itself, and the compatibility.) A smooth submanifold S of M should satisfy three conditions:

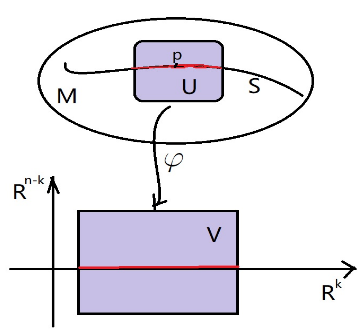
- S should be a subset of M;
- S itself should be a smooth manifold of dimension $k \leq n$;
- the smooth structures on S and on M should be compatible.

The last condition, i.e. the compatibility, can be stated more precisely: the smooth structure (=a set of coordinate charts) on S should be the "restriction" of the smooth structure on M.

Definition 1.1. A subset $S \subset M$ is a k-dimensional *smooth submanifold* of M if for every $p \in S$, there is a chart (φ, U, V) around p of M such that

$$\varphi(U \cap S) = V \cap (\mathbb{R}^k \times \{0\}) = \{x \in \varphi(U) \mid x^{k+1} = \cdots = x^n = 0\}.$$

We will call $\operatorname{codim}(S) = n - k$ the *codimension* of S.



Remark. Roughly speaking, smooth submanifolds are objects that are defined locally by equations

$$\varphi_{k+1} = \cdots = \varphi_n = 0.$$

Note that $\varphi_{k+1}, \cdots, \varphi_n$ are smooth functions on U, since φ is a diffeomorphism.

Example. Let M, N be smooth manifolds, and $f : M \to N$ is a smooth map. Then the graph

$$\Gamma_f = \{(p, q) \mid q = f(p)\}$$

is a smooth submanifold of $M \times N$. To see this, we take a chart (φ, U, V) of M near p and a chart (ψ, X, Y) of N near $q = f(p)$. Then (c.f. PSet 2 Problem 5) $(\varphi \times \psi, U \times X, V \times Y)$ is a chart of $M \times N$ near (p, q). But this chart is not good for our purpose. To get a chart that is suitable for our purpose, we write the equation $q = f(p)$ as $\psi^{-1}(y) = f(\varphi^{-1}(x))$, i.e. $y = \psi(f(\varphi^{-1}(x)))$. Now we define a smooth map

$$\Psi : V \times Y \to \mathbb{R}^m \times \mathbb{R}^n, \quad (a, b) \mapsto (a, b - \psi \circ f \circ \varphi^{-1}(a)).$$

It is easy to see that Ψ is one-to-one and is a local diffeomorphism everywhere, thus is a global diffeomorphism from $V \times Y$ onto its image $\Psi(V \times Y)$, which is an Euclidian open set. Thus $(\Psi \circ (\varphi \times \psi), U \times X, \Psi(V \times Y))$ is also a local chart of $M \times N$ near (p, q). Moreover, with this local chart,

$$(p, q) \in \Gamma_f \cap (U \times X) \Longrightarrow \psi(q) = \psi(f(\varphi^{-1}(\varphi(p)))) \Longrightarrow \Psi(\varphi(p), \psi(q)) = (\varphi(p), 0),$$

so the conclusion follows.

Remark. In Lecture 2 we mentioned that the graph Γ_f of any continuous function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a smooth manifold of dimension n by itself. It is of course a subset of $\mathbb{R}^{n+1}$. However, in general Γ_f is not a smooth submanifold of $\mathbb{R}^{n+1}$ if f is not smooth. (If you repeat the arguments in the above example, which step does not work?)(However, in PSet 3 Part 1 you will see an example where f is not a smooth function on the whole domain, while Γ_f is a smooth submanifold.)

Example. The sphere S^n is a smooth submanifold of $\mathbb{R}^{n+1}$. Can you construct a local chart of $\mathbb{R}^{n+1}$ near every point of S^n which satisfies the condition in the definition 1.1?

Note that in the definition of a smooth submanifold above, we did not spell out the smooth structure on S. To construct natural charts on S, we denote

$$\pi : \mathbb{R}^n \to \mathbb{R}^k, \quad (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^k)$$
$$\jmath : \mathbb{R}^k \hookrightarrow \mathbb{R}^n, \quad (x^1, \cdots, x^k) \mapsto (x^1, \cdots, x^k, 0, \cdots, 0).$$

Then we have

Proposition 1.2. *Let (φ, U, V) be a chart on M that satisfies Definition 1.1. Let $X = U \cap S$, $Y = \pi \circ \varphi(X)$ and $\psi = \pi \circ \varphi|_X$. Then (ψ, X, Y) is a smooth chart on S.*

Proof. By definition, ψ is invertible and the inverse $\psi^{-1} = \varphi^{-1} \circ \jmath$. So (ψ, X, Y) is a chart on S. It remains to check that charts of this type are compatible. In fact, the transition maps are

$$\psi_\beta \circ \psi_\alpha^{-1} = \pi \circ \varphi_\beta \circ \varphi_\alpha^{-1} \circ \jmath = \pi \circ \varphi_{\alpha,\beta} \circ \jmath,$$

which are smooth. $\square$

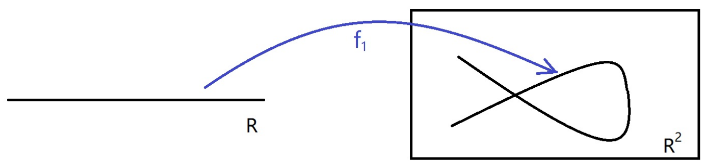
So S is a smooth manifold by itself. Now consider the inclusion map $\iota : S \hookrightarrow M$. With respect to the smooth structures described above, ι is a smooth immersion since

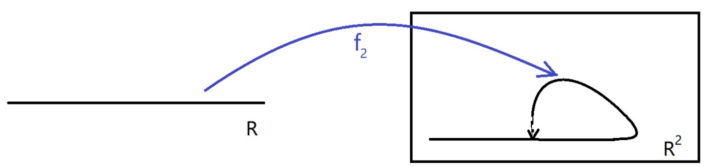
$$\varphi \circ \iota \circ \psi^{-1} = \jmath.$$

So any smooth submanifold is the image of a smooth immersion.

It is natural to ask whether the image of any smooth immersion is a smooth submanifold. Unfortunately this is not true in general:

Example. The following two graphs are the images of two immersions of $\mathbb{R}$ into $\mathbb{R}^2$.





For the first one, the immersion is not injective. For the second one, the immersion is injective, while the image still have different topology than $\mathbb{R}$.

One can construct an even more complicated example: consider $f_3 : \mathbb{R} \to \mathbb{T}^2 = S^1 \times S^1$ defined by

$$f(t) = (e^{it}, e^{i\sqrt{2}t}).$$

Then f is an immersion, and the image $f(\mathbb{R})$ is a "dense curve" in the torus $S^1 \times S^1$.

Remark. For the three "immersion" above whose image are not submanifolds, the first one is "worst" since the image is not a manifold in any sense: at the crossing point, the image is not a manifold, no matter what topology you give to the image. On the other hand, for the second one and the third one, we can easily see that

- if we use the "subspace topology" inherited from $\mathbb{R}^2$ or $\mathbb{T}^2$, then the images are not manifolds;
- if we endow the images with the topology that "borrowed" from $\mathbb{R}$, then the images are smooth manifolds!

In general, the image of any injective immersion is a manifold, where the manifold structure is "borrowed" from the source manifold. So people call the images of injective immersions *immersed submanifolds*. To distinguish immersed submanifolds with smooth submanifolds defined in Definition 1.1, sometimes people call smooth submanifolds *embedded submanifolds* or *regular submanifolds*.

2. Embeddings

What is the difference between smooth submanifolds and immersed submanifolds? As we just described, the topology that makes an immersed submanifold a manifold is the topology from the source manifold, not the "subspace topology" from the target manifold. On the other hand, if S is a smooth submanifold of M, then the topology underlying the smooth manifold S is the topology generated by charts (ψ, X, Y) in Proposition 1.2. By definition 1.1 it is easy to see

Proposition 2.1. *Let S be a smooth submanifold of M. Then $\iota : S \hookrightarrow M$ is a homeomorphism from S to $\iota(S)$ (endowed with the subspace topology from M).*

Remark. If S is a smooth submanifold of M, then there is a unique topology/smooth structure on S so that the inclusion map $\iota : S \hookrightarrow M$ is a smooth immersion which is a homeomorphism onto its image. (See Theorem 5.31 in Lee's book.)

So for any smooth submanifold S, the inclusion map $\iota : S \hookrightarrow M$ is a special immersion which is a homeomorphism onto its image.

Definition 2.2. Let M, N be smooth manifolds, and $f : N \to M$ an immersion. f is called an *embedding* if it is a homeomorphism onto its image $f(N)$, where the topology on $f(N)$ is the subspace topology as a subset of M.

By definition, the inclusion map $\iota : S \hookrightarrow M$ is an embedding. So

$$\boxed{\text{each smooth submanifold is the image of an embedding.}}$$

Conversely,

Theorem 2.3. *Let $f : N \to M$ be an embedding. Then the image $f(N)$ is a smooth submanifold of M.*

Proof. Let $p \in N$ and $q = f(p)$. Since f is an immersion, the canonical immersion theorem implies that there exists charts (φ_1, U_1, V_1) near p and (ψ_1, X_1, Y_1) near q such that on V_1, $\psi_1 \circ f \circ \varphi_1^{-1}$ is the canonical embedding $\jmath : \mathbb{R}^m \to \mathbb{R}^n$ restricted to V_1, i.e.

$$\psi_1 \circ f = \jmath \circ \varphi_1$$

on U_1. Since f is a homeomorphism onto its image, $f(U_1)$ is open in $f(N) \subset M$. In other words, there exists an open set $X \subset M$ such that $f(U_1) = f(N) \cap X$. Replace X_1 by $X_1 \cap X$, and Y_1 by $\psi_1(X_1 \cap X)$. Then for this new chart (ψ_1, X_1, Y_1),

$$\psi_1(X_1 \cap f(N)) = Y_1 \cap \psi_1(f(U_1)) = Y_1 \cap \jmath(\varphi_1(U_1)) = Y_1 \cap (\mathbb{R}^m \times \{0\}).$$

□

Please repeat the above argument on the "dense curve in $\mathbb{T}^2$" example to see what's wrong there.

Here is the main difference between an immersion and an embedding:

- If $f : N \to M$ is an immersion, then by the canonical immersion theorem, any point $p \in N$ has a neighborhood in N whose image is "nice" in M.

- If $f : N \to M$ is an embedding, then by Theorem 2.3, any point $q \in f(N)$ has a neighborhood in $f(N)$ that is "nice" in M.

In general an immersion need not be an embedding. However, if N is compact, then

Theorem 2.4. *If $f : N \to M$ is an injective immersion, and N is compact, then f is an embedding.*

Proof. Since f is injective, $f : N \to f(N)$ is invertible. Since $f : N \to M$ is continuous, $f : N \to f(N)$ is also continuous. It remains to show that $f^{-1} : f(N) \to N$ is continuous. To prove this, we take an arbitrary closed set A in N. Then since N is compact, A has to be compact. So $(f^{-1})^{-1}(A) = f(A)$ is a compact since f is continuous. Because $f(N)$ is Hausdorff, $f(A)$ is a closed set in $f(N)$. In other words, $(f^{-1})^{-1}(A)$ is closed as long as A is closed. So f^{-1} is continuous. □

As a consequence of this theorem, we can prove that any compact smooth manifold can be realized as a submanifold of the Euclidian space:

Theorem 2.5 (The Whitney embedding theorem: the easiest version)**.** *Any compact smooth manifold M can be embedded into $\mathbb{R}^K$ for sufficiently large K.*

Here is the idea: For any local chart $\{\varphi, U, V\}$, φ is an embedding from U into the Euclidian space of the same dimension. If M is compact, then M can be covered by finitely many charts. Although φ's are only defined on U's, one can multiply φ's by functions in a P.O.U. so that they are defined on M. It turns out that this idea almost work.

Next time we will prove the theorem. We will also remove the compactness assumption in the theorem, and show that the dimension K could be chosen to be $2n + 1$.