LECTURE 10: TUBULAR NEIGHBORHOOD THEOREM

1. Generalized Inverse Function Theorem

We will start with

Theorem 1.1 (Generalized Inverse Function Theorem, compact version)**.** *Let $f : M \to N$ be a smooth map that is one-to-one on a compact submanifold X of M. Moreover, suppose $df_x : T_xM \to T_{f(x)}N$ is a linear diffeomorphism for each $x \in X$. Then f maps a neighborhood U of X in M diffeomorphically onto a neighborhood V of $f(X)$ in N.*

Note that if we take $X = \{x\}$, i.e. a one-point set, then is the inverse function theorem we discussed in Lecture 2 and Lecture 6. According to that theorem, one can easily construct neighborhoods U of X and V of $f(X)$ so that f is a local diffeomorphism from U to V. To get a global diffeomorphism from a local diffeomorphism, we will need the following useful lemma:

Lemma 1.2. *Suppose $f : M \to N$ is a local diffeomorphism near every $x \in M$. If f is invertible, then f is a global diffeomorphism.*

Proof. We need to show f^{-1} is smooth. Fix any $y = f(x)$. The smoothness of f^{-1} at y depends only on the behaviour of f^{-1} near y. Since f is a diffeomorphism from a neighborhood of x onto a neighborhood of y, f^{-1} is smooth at y. □

Proof of Generalized IFT, compact version. According to Lemma 1.2, it is enough to prove that f is one-to-one in a neighborhood of X. We may embed M into $\mathbb{R}^K$, and consider the "ε-neighborhood" of X in M:

$$X^\varepsilon = \{x \in M \mid d(x, X) < \varepsilon\},$$

where $d(\cdot, \cdot)$ is the Euclidean distance. Note that X^ε is a bounded set in $\mathbb{R}^K$ since X is compact. And we have $X = \bigcap_{k>0} X^{1/k}$, again since X is compact.

If f is not one-to-one on each $X^{1/k}$, then one can find $a_k \neq b_k \in X^{1/k}$ such that $f(a_k) = f(b_k)$. Since all a_k's lie in a bounded closed set in $\mathbb{R}^K$, which is compact, one can find a subsequence such that $a_{k_i} \to a_\infty \in X$. Similarly there is a subsequence $b_{k_{i_j}} \to b_\infty \in X$. Since $f(a_\infty) = f(b_\infty)$, one must have $a_\infty = b_\infty$, since f is one-to-one on X. So in any neighborhood of a_∞, f is not one-to-one. But df_{a_∞} is linear isomorphism implies that f is a local diffeomorphism near a_∞, which is a contradiction. □

By staring at the proof, one can see that in the proof, we only need to require that X is a <u>compact subset</u> in M. So we in fact proved

Theorem 1.3 (Generalized Inverse Function Theorem, compact subset version)**.** *Let $f : M \to N$ be a smooth map that is one-to-one on a compact subset X of M. Moreover, suppose $df_x : T_xM \to T_{f(x)}N$ is a linear diffeomorphism for each $x \in X$. Then f maps a neighborhood U of X in M diffeomorphically onto a neighborhood V of $f(X)$ in N.*

It is natural to ask: what if we remove the "compactness" assumption? In general, the theorem is not true, even if we assume that X is a smooth submanifold. Here is a simple example:

Example. Consider the map

$$f : \mathbb{R}^2 \to \mathbb{T}^2, \quad (t, s) \mapsto (e^{it}, e^{is}).$$

It is easy to see that f is a local diffeomorphism near any $(t, s) \in \mathbb{R}^2$. However, if we take X be the "irrational-slope line"

$$X = \{(t, \sqrt{2}t) \mid t \in \mathbb{R}\}$$

in $\mathbb{R}^2$ (which is a perfectly nice smooth submanifold), then there is no hope to find neighborhoods U of X in $\mathbb{R}^2$ and V of $f(X)$ in $\mathbb{T}^2$ so that f maps U diffeomorphically onto V, since $f(X)$ is dense in $\mathbb{T}^2$.

Note that if X is a compact submanifold of X, then $f(X)$ is in fact a compact smooth submanifold of N (since $f|_X : X \to N$ is an embedding), and f is a diffeomorphism from X onto $f(X)$. In the example above, the bad thing is that the image $f(X)$ of X is not a smooth submanifold of $\mathbb{T}^2$ (but only an immersed submanifold), and thus f is not a diffeomorphism from X onto $f(X)$ (with subspace topology from the target).

It turns out that if we require $f(X)$ to be a smooth submanifold of N, and require $f : X \to f(X)$ to be a diffeomorphism, then we still have the generalized inverse function theorem near X:

Theorem 1.4 (Generalized Inverse Function Theorem, non-compact version)**.** *Let $f : M \to N$ be a smooth map that is one-to-one on a smooth submanifold X of M. Moreover, suppose f maps X diffeomorphically onto $f(X)$ (which is assumed to be a smooth submanifold in N), and suppose $df_x : T_xM \to T_{f(x)}N$ is a linear diffeomorphism for each $x \in X$. Then f maps a neighborhood U of X in M diffeomorphically onto a neighborhood V of $f(X)$ in N.*

Proof. The idea is standard: Any non-compact manifold can be written as a union of many "compact stripes". By using any positive smooth exhaustion function g on $f(X)$, we may decompose $f(X) = \bigcup_{k=1}^{\infty} K_k$, where $K_k = g^{-1}([k, k+1])$. Since $f|_X : X \to f(X)$ is a diffeomorphism, each $J_k := f|_X^{-1}(K_k)$ is compact in X and give such a decomposition for X. In particular, by Theorem 1.3, there is an open neighborhood $\widetilde{U}_k$ of $J_{k-1} \cup J_k \cup J_{k+1}$ in M, such that f is a diffeomorphism on $\widetilde{U}_k$. Since $f(X)$ is a smooth submanifold in N, by embedding N into $\mathbb{R}^K$ and using the induced distance function, one may prove $d_k := \operatorname{dist}(K_k, \bigcup_{j>k+1} K_j) > 0$ (Here we need to use the compactness of K_k and use the fact that $f(X)$ is a smooth submanifold).

Now we choose a decreasing sequence of positive numbers $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that $\varepsilon_k < d_k/2$ for each k. Note that this implies $K_l^{\varepsilon_l} \cap K_k^{\varepsilon_k} = \emptyset$ if $|l-k| > 1$. Define

$$U_k = \widetilde{U}_{k-1} \cap \widetilde{U}_k \cap \widetilde{U}_{k+1} \cap f^{-1}(K_k^{\varepsilon_k}).$$

Then U_k is an open neighborhood of J_k, $V_k = f(U_k)$ is an open neighborhood of K_k, and f maps U_k diffeomorphically onto V_k. Moreover, the definition of U_k implies $U_{k-1} \cup U_k \cup U_{k+1} \subset \widetilde{U}_k$.

We define $U = \bigcup_{k\geq 1} U_k$ and $V = \bigcup_{k\geq 1} V_k$. Then U is an open neighborhood of X in M, V is an open neighborhood of $f(X)$ in N, and $f : U \to V$ is a local diffeomorphism everywhere. It remains to show f is injective. Suppose $x, y \in U$ and $f(x) = f(y)$. Then there exists k such that $f(x) = f(y) \in V_k \subset K_k^{\varepsilon_k}$. It follows $x, y \in U_{k-1} \cup U_k \cup U_{k+1} \subset \widetilde{U}_k$. Since f is a diffeomorphism on $\widetilde{U}_k$, we conclude $x = y$. $\square$

2. Tubular Neighborhood Theorem

Now let $X \subset M$ be a smooth submanifold. It is natural to ask: What does M looks like "near" X? The famous tubular neighborhood theorem claims that X always admits a "tubular" neighborhood inside M. Moreover, the tubular neighborhood looks like a neighborhood of X inside its "normal bundle". This gives some kind of "canonical form" of a neighborhood of any smooth submanifold.

We first prove the following "Euclidean version" of the tubular neighborhood theorem, which gives us a nice neighborhood of any submanifold in $\mathbb{R}^K$:

Theorem 2.1 (ε-Neighborhood Theorem)**.** *Let $\iota : X \to \mathbb{R}^K$ be a smooth submanifold. Then there exists a smooth positive function $\varepsilon : X \to \mathbb{R}^+$, such that if we let X^ε be the ε-neighborhood of X,*

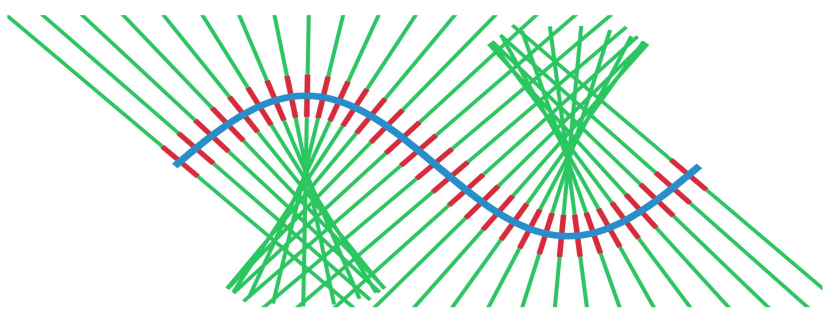
$$X^\varepsilon := \{y \in \mathbb{R}^K \mid |y - x| < \varepsilon(x) \text{ for some } x \in X\},$$

then

(1) *each $y \in X^\varepsilon$ possesses a unique closest point $\pi_\varepsilon(y)$ in X;*
(2) *the map $\pi_\epsilon : X^\varepsilon \to X$ is a submersion.*

(Note: If X is compact, then ε can be taken to be a constant.)

The ε-neighborhood described in the above theorem looks like



We will prove the theorem by constructing a diffeomorphism between a neighborhood of X inside its *normal bundle* (which can be think of as a nice "non-intersecting"

way to put the green lines above together to form a smooth manifold) and a neighborhood of X inside $\mathbb{R}^K$. So let's first recall the conception of the normal bundle of a manifold embedded in $\mathbb{R}^K$, which is defined in PSet 3 Part 1 Problem 4. Let $\iota : X \hookrightarrow \mathbb{R}^K$ be a smooth submanifold of dimension m. For each $x \in X$, we can identify T_xX with an m-dimensional vector subspace in $\mathbb{R}^K$ via

$$T_xX \simeq d\iota_x(T_xX) \subset T_x\mathbb{R}^K \simeq \mathbb{R}^K.$$

Let N_xX be the orthogonal complement of T_xX in $\mathbb{R}^K$,

$$N_xX := \{v \in T_x\mathbb{R}^K \simeq \mathbb{R}^K \mid v \perp T_xX\},$$

which is a $(K-m)$-dimensional vector subspace. Define

$$NX = \{(x, v) \in \mathbb{R}^K \times \mathbb{R}^K \mid x \in X, v \in N_xX\} \subset T\mathbb{R}^K.$$

In PSet 3 Part 1 Problem 4 you are supposed to show that NX is a K-dimensional smooth submanifold in $T\mathbb{R}^K$, called the *normal bundle* of X. Moreover, the canonical projection map

$$\pi : NX \to X, \quad (x, v) \mapsto x$$

is a submersion.

Remark. The conception of normal bundle NX is extrinsic: it depends on the ambient space $\mathbb{R}^K$ and also depends on the way of embedding $\iota : M \to \mathbb{R}^K$.

Proof of the ε-Neighborhood Theorem. Define a map

$$h : NX \to \mathbb{R}^K, \quad (x, v) \mapsto x + v.$$

Then at any point $(x, 0) \in NX$, dh is non-singular, because $dh_{(x,0)}$ maps $T_{(x,0)}(X \times \{0\}) \subset T_{(x,0)}NX$ bijectively onto $T_xX \subset T_x\mathbb{R}^K$, and maps the tangent space of $T_{(x,0)}(\{x\} \times N_xX)$ bijectively onto $N_xX \subset T_x\mathbb{R}^K$.

Also by definition, h maps $X \times \{0\} \subset NX$ diffeomorphically onto $X \subset \mathbb{R}^K$. According to the generalized Inverse Function Theorem above, h maps a neighborhood U of $X \times \{0\}$ in NX diffeomorphically onto a neighborhood V of X in $\mathbb{R}^K$.

To show that there exists $\varepsilon \in C^\infty(X)$ such that $X^\varepsilon \subset V$, for each $x \in X$ we let $\tilde{\varepsilon}(x) = \sup\{r \le 1 \mid B_r(x) \subset V\}$. Then $\tilde{\varepsilon}$ is a positive continuous function on X. (Can you prove the continuity of $\tilde{\varepsilon}$?) By using Whitney approximation theorem for functions (See Lecture 4), we can find positive $\varepsilon \in C^\infty(X)$ such that $|\varepsilon - \tilde{\varepsilon}/2| < \tilde{\varepsilon}/4$. In particular we have $\varepsilon(x) < \tilde{\varepsilon}(x)$. It follows that for such ε, $X^\varepsilon \subset V$.

Now we define

$$\pi_\varepsilon : X^\varepsilon \to X, \qquad y \mapsto \pi_\varepsilon(y) = \pi \circ h^{-1}(y).$$

It is a submersion since π is a submersion and h^{-1} is a diffeomorphism on V. It remains to prove that $\pi_\varepsilon(y)$ is the unique closest point to y in X. In fact, let $z \in X$ be the closest point in X to y. Then the sphere centered at y with radius $|y - z|$ is tangent to X at z. It follows that the vector $y - z$ is perpendicular to X at z, i.e. $y - z \in N_zX$. So we have $y = z + (y - z) = h(z, y - z)$, i.e. $\pi_\varepsilon(y) = z$. This completes the proof. $\square$

In general, if $X \subset M$ is a smooth submanifold. Then one can still define the normal bundle $N(X, M)$ (with respect to the ambient space M) as the set of points of the form (x, v), where for any $x \in X$, v is any vector in the quotient space

$$N_x(X, M) = T_xM/T_xX.$$

One can show that $N(X, M)$ is a smooth manifold whose dimension equals $\dim M$. .

To get a more extrinsic (and geometric) description of $N(X, M)$, we may embed M into $\mathbb{R}^K$. Then we will get an inclusion $T_xX \subset T_xM \subset T_x\mathbb{R}^K$, and the quotient space T_xM/T_xX can be identified as the space of vectors in T_xM which are perpendicular to T_xX. It follows

$$N(X, M) \simeq \{(x, v) \mid x \in X, v \in T_xM \text{ and } v \perp T_xX\}.$$

Note that by using this identification, we have

$$T_{(x,0)}N(X, M) \simeq T_xX \oplus T_x^{\perp}X,$$

where $T_x^{\perp}X$ is the orthogonal complement of T_xX inside T_xM.

Now we prove

Theorem 2.2 (Tubular Neighborhood Theorem)**.** *Let $X \subset M$ be a smooth submanifold. Then there exists a diffeomorphism from an open neighborhood of X in $N(X, M)$ onto an open neighborhood of X in M.*

Proof. Embed M into $\mathbb{R}^K$. Let $\pi_\varepsilon : M^\varepsilon \to M$ be as in the ε-neighborhood theorem (for the embedding $\iota : M \hookrightarrow \mathbb{R}^K$). Again consider the map

$$h : N(X, M) \to \mathbb{R}^K, \quad h(x, v) \to x + v.$$

Then $W := h^{-1}(M^\varepsilon)$ is an open neighborhood of X in $N(X, M)$. Now consider the composition

$$h_\varepsilon = \pi_\varepsilon \circ h : W \longrightarrow M.$$

Then h_ε is smooth, and is the identity map on $X \subset N(X, M)$. Moreover, according to the decomposition of $T_{(x,0)}N(X, M)$ above, $(dh_\varepsilon)_{(x,0)}$ maps $T_{(x,0)}N(X, M)$ bijectively onto T_xM. So the theorem follows from the generalized IFT. □

3. Smooth Approximations of Continuous Maps

Recall that the Whitney approximation theorem for functions in Lecture 4 tells us that one can approximate any continuous function by a smooth function in arbitrary accuracy. Now let M, N be smooth manifolds and let $g : M \to N$ be a continuous map. A natural question to ask is: Can we "approximate" g by a smooth map $f : M \to N$?

Of course there is one issue in proposing the above question: what do we mean by "f approximate g"? If N is $\mathbb{R}^K$, one can require $|f - g| < \varepsilon$ which is exactly what we did in Lecture 4. But in the general case, $f(x)$ and $g(x)$ are points in a smooth manifold N, and it makes no sense to talk about $f(x) - g(x)$. (Of course one can embed N into $\mathbb{R}^K$. But then $|f - g|$ will depend on the way of embedding, and thus is not a good way of measuring whether f and g are "close", since f and g themselves has noting to do with the way of embedding N into $\mathbb{R}^K$.)

In topology, we do have one way to talk about the relation between two maps, which is very useful in many contexts. That is the conception of homotopy. Roughly speaking, we say two maps f_0 and f_1 are homotopic if one can continuously deform f_0 to f_1. More precisely,

Definition 3.1. Let X, Y be topological spaces. We say $f_0, f_1 \in C^0(X, Y)$ are *homotopic* if there exists $F \in C^0(X \times [0,1], Y)$ so that

$$F(\cdot, 0) = f_0(\cdot) \text{ and } F(\cdot, 1) = f_1(\cdot).$$

Now we prove the Whitney Approximation Theorem for maps, which claims that any continuous map between smooth manifolds can be continuously deformed to a smooth map:

Theorem 3.2 (Whitney Approximation Theorem for Continuous Maps)**.**
Given any continuous mapping $g \in C^0(M, N)$, one can find a smooth mapping $f \in C^\infty(M, N)$ which is homotopic to g. Moreover, if g is smooth on a closed subset $A \subset M$, then one can choose f so that $f = g$ on A.

Proof. We embed N into $\mathbb{R}^K$, and consider N^ε, the ε-neighborhood of N in $\mathbb{R}^K$. Since ε is a positive smooth function on N, the composition $\overline{\varepsilon} = \varepsilon \circ g$ is a positive continuous function on M. Think of g as a continuous function from M to $\mathbb{R}^N$. According to the Whitney approximation theorem in Lecture 4, there is a smooth function $\tilde{f} : M \to \mathbb{R}^K$ which is ε-close to g, i.e.

$$|\tilde{f}(x) - g(x)| < \varepsilon(g(x)), \quad \forall x \in M.$$

So $\tilde{f}(x) \in B(g(x), \varepsilon(g(x)))$. It follows that

$$(1-t)g(x) + t\tilde{f}(x) \in B(g(x), \varepsilon(g(x))) \subset N^\varepsilon, \quad \forall\ 0 \le t \le 1.$$

Now define $F : M \times [0,1] \to N$ by

$$F(x,t) = \pi_\varepsilon((1-t)g(x) + t\tilde{f}(x)).$$

Then F is a homotopy that connects the continuous map g to the smooth map

$$f = F(\cdot, 1) = \pi_\varepsilon \circ \tilde{f} : M \to N.$$

Finally note that if g is smooth on a closed subset A, then the smooth function $\tilde{f}$ can be chosen so that $\tilde{f} = g$ on A. It follows that $f = g = F(\cdot, t)$ on A. (In other words, the homotopy connecting g to f can be chosen to be relative to A.) □

Since in this course, we are mainly interested in smooth objects (smooth manifolds, smooth submanifolds, smooth functions, smooth maps, smooth vector fields, smooth vector bundle, smooth differential form etc), we are interested in homotopies connecting two smooth maps via "smooth path", i.e.

Definition 3.3. We say $f_0, f_1 \in C^\infty(M, N)$ are *smoothly homotopic* if there exists $F \in C^\infty(M \times [0,1], N)$ so that

$$F(\cdot, 0) = f_0 \text{ and } F(\cdot, 1) = f_1.$$

Of course if f_0 and f_1 are smoothly homotopic, then they are homotopic. Conversely, we have

Theorem 3.4 (Homotopy $\equiv$ Smooth homotopy)**.**
Suppose $f_0, f_1 \in C^\infty(M, N)$ are homotopic, then they are smoothly homotopic.

Proof. Let $F : M \times [0,1] \to N$ be a homotopy connecting f_0 and f_1. Continuously extend F to a mapping $\widetilde{F} : M \times \mathbb{R} \to N$ by defining

$$\widetilde{F}(x,t) = F(x,0) \text{ if } t \leq 0, \quad \text{and} \quad \widetilde{F}(x,t) = F(x,1) \text{ if } t \geq 1.$$

Then $\widetilde{F}$ is a continuous map from $M \times \mathbb{R}$ to N, and is smooth on closed subsets $M \times \{0\}$ and $M \times \{1\}$. (Recall that by definition in Lecture 4, $\tilde{F}$ is smooth on $M \times \{0\}$ means there is a smooth function $\tilde{G}$ defined on $M \times (-\varepsilon, \varepsilon)$ so that $\tilde{G}(x,0) = \tilde{F}(x,0)$. We don't require $\tilde{F}$ to be smooth in a neighborhood of $M \times \{0\}$.) So by Whitney Approximation Theorem for maps above, there exists a smooth map $\overline{F} : M \times \mathbb{R} \to N$ (that is homotopic to $\widetilde{F}$, which we don't need here), such that $\overline{F} = \widetilde{F}$ on $M \times \{0\}$ and $M \times \{1\}$, i.e.

$$\overline{F}(\cdot, 0) = f_0 \text{ and } \overline{F}(\cdot, 1) = f_1.$$

It follows that $\overline{F}$ is the desired smooth homotopy connecting f_0 and f_1. $\square$

Note that homotopy is an equivalence relation on the space of continuous maps from M to N. The equivalence classes are called homotopy classes of maps. Theorem 3.4 implies that smooth homotopy is an equivalence relation on the space of smooth maps from M to N. Moreover, combining Theorem 3.2 and Theorem 3.4, we immediately see that homotopy classes of continuous maps coincides with the smooth homotopy classes of smooth maps.