LECTURE 11: TRANSVERSALITY

Let $f : M \to N$ be a smooth map. In the past three lectures, we are mainly studying the image of f, especially when f is an embedding. Today we would like to study the pre-image of a smooth map f. In particular, we would like to study the following question: when will the pre-image of a nice submanifold in N be a smooth submanifold in M?

A special local model of the problem is as follows. Recall that a submanifold X inside a smooth manifold M is, at least locally in small coordinate charts in M, defined by a set of equations of the form $\varphi_{k+1} = \cdots = \varphi_n = 0$, where each φ_i is a (locally defined) smooth function. In other words, X is (locally) the 0-level set of the smooth map $g = (\varphi_{k+1}, \cdots, \varphi_n)$, or the pre-image of the submanifold $\mathbb{R}^k \times \{0\}$ under the smooth map $\varphi = (\varphi_1, \cdots, \varphi_n)$.

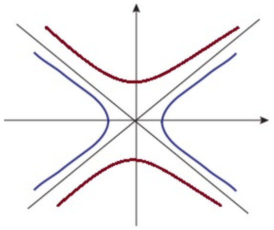
1. Smooth submanifolds as level sets

Let $f : M \to N$ be a smooth map. Let $q \in N$. We would ask: when will the level set $f^{-1}(q)$ a smooth submanifold in M?

Here is a very simple example. Consider the map

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 - y^2.$$

Obviously for any $t \neq 0$, $f^{-1}(t)$ is a smooth manifold in $\mathbb{R}^2$ (which is a hyperbola), but $f^{-1}(0)$ consist of two "crossing lines" and thus is not a smooth manifold at all.



Why the value 0 is special? Well, as a 1×2 matrix,

$$df = (2x, -2y).$$

So the only critical point of f is the origin $(0, 0)$. It follows that $0 = f(0, 0)$ is the only critical value of f!

In general, we have

Theorem 1.1. *If q is a regular value of a smooth map $f : M \to N$, then $S = f^{-1}(q)$ is a smooth submanifold of M of dimension $\dim M - \dim N$. Moreover,*

$$T_pS = \ker(df_p : T_pM \to T_qN), \qquad \forall p \in S,$$

where we have identified T_pS with $d\iota_p(T_pS) \subset T_pM$.

Proof. Let $p \in S = f^{-1}(q)$. Then by the canonical submersion theorem, there are charts (φ_1, U_1, V_1) centered at p and (ψ_1, X_1, Y_1) centered at q such that $f(U_1) \subset X_1$, and $\pi = \psi_1 \circ f \circ \varphi_1^{-1}$. It follows that φ_1 maps $U_1 \cap f^{-1}(q)$ onto $V_1 \cap \pi^{-1}(0)$. Since $\pi^{-1}(0) = \{0\} \times \mathbb{R}^k$, where $k = \dim M - \dim N$, the first part of the theorem follows from the definition of smooth submanifolds.

Now denote the inclusion by $\iota : S \hookrightarrow M$. Then for any $p \in S$, $f \circ \iota(p) = q$. In other words, $f \circ \iota$ is a constant map on S. So $df_p \circ d\iota_p = 0$, i.e. $df_p = 0$ on the image of $d\iota_p : T_pS \hookrightarrow T_pM$, or in other words,

$$T_pS \subset \ker(df_p).$$

Finally by dimension counting,

$$\dim T_pS = \dim S = \dim M - \dim N = \dim T_pM - \dim T_qN = \dim \ker(df_p),$$

we conclude that $T_pS = \ker(df_p)$. □

Since most points in N are regular values of f, in most cases we get many smooth submanifolds in M. Theorem 1.1 is in fact one of the most powerful tool to construct smooth manifolds.

Example. Consider the map

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \quad (x^1, \cdots, x^{n+1}) \mapsto (x^1)^2 + \cdots + (x^{n+1})^2.$$

Then $S^n = f^{-1}(1)$. Since the differential

$$df_a = 2(a^1, \cdots, a^{n+1}) \neq 0$$

for any $a \in S^n$, we conclude that S^n is an n-dimensional submanifold of $\mathbb{R}^{n+1}$. Moreover, for any $a = (a^1, \cdots, a^{n+1}) \in S^n$,

$$T_aS^n = \ker(df_a) = \{v \in \mathbb{R}^{n+1} \mid v \cdot a = 0\},$$

which coincides with our usual understanding of the tangent space of the sphere.

2. Transversal intersection

Now let $f : M \to N$ and let $X \subset N$ be a smooth submanifold. We consider the following question:

$$\boxed{\text{When will } f^{-1}(X) \text{ be a smooth submanifold in } M?}$$

Note that "whether $f^{-1}(X)$ is a smooth submanifold in M or not" is a local problem: it is enough to examine, for each $p \in f^{-1}(X)$, whether there is a neighborhood U of p in M so that $U \cap f^{-1}(X)$ is a smooth submanifold of M. Since X is a smooth submanifold of N, locally in a neighborhood V of $q = f(p)$ in N, one can find a smooth function $g : V \to \mathbb{R}^l$, where $l = \dim N - \dim X$ is the codimension of X in N, such that $g^{-1}(0) = X$. (In the language of Lec. 8, g can be taking to be the map $g = (\varphi_{k+1}, \cdots, \varphi_n)$.) As a consequence, we can write $f^{-1}(X)$ as the level set of the composition $g \circ f$:

$$f^{-1}(X) = f^{-1} \circ g^{-1}(0) = (g \circ f)^{-1}(0).$$

So according to Theorem 1.1, for $f^{-1}(X)$ to be a smooth submanifold of M, it is enough to check whether 0 is a regular value of $g \circ f$.

Now we would like to find conditions on f and S so that 0 is a regular value of $g \circ f$. (Since given f and X, the map g is not unique, we would like to find conditions that is independent of g.) In other words, we need to figure out conditions so that for any $p \in (g \circ f)^{-1}(0) = f^{-1}(X)$, $d(g \circ f)_p$ is surjective. By chain rule,

$$d(g \circ f)_p = dg_q \circ df_p.$$

Note that by definition, dg_q is already surjective, with kernel $\ker(dg_q) = T_qX$. So for $d(g \circ f)_p$ to be surjective, it is enough to assume that $\mathrm{Im}(df_p)$ contains some "complementary subspace" of T_qX in T_qN.

To be more precise, we will need the following linear algebra lemma:

Lemma 2.1. *Let $L : V \to W$ be any surjective linear map, and $V_1 \subset V$ be any vector subspace. Then $L(V_1) = W$ if and only if $V_1 + \ker(L) = V$.*

Proof. If $V_1 + \ker(L) = V$, then $L(V_1) = L(V_1 + \ker(L)) = L(V) = W$.

If $V_1 + \ker(L) \neq V$, we pick any $v \in V$ but $v \notin V_1 + \ker(L)$. Then $L(v) \notin L(V_1)$, since if there is $v_1 \in V_1$ such that $L(v_1) = L(v)$, then $v - v_1 \in \ker(L)$, i.e.

$$v = v_1 + (v - v_1) \in V_1 + \ker(L),$$

a contradiction. It follows that $L(V_1) \neq W$. □

According to this lemma, 0 is a regular value of $g \circ f$ if and only if

$$\mathrm{Im}(df_p) + T_{f(p)}X = T_{f(p)}N, \qquad \forall p \in f^{-1}(X). \tag{1}$$

Note that this condition depends only on f and X, and does not depend on g.

Definition 2.2. Let $f : M \to N$ be a smooth map, and $X \subset N$ be a smooth submanifold. We say f intersect X *transversally*, and denote by $f \pitchfork X$, if (1) holds.

We remark that if $f(M) \cap X = \emptyset$, i.e. if $f^{-1}(X) = \emptyset$, then f intersects X transversally, since there is no condition to check at all.

As another special case, we may consider a smooth map $f : M \to N$ and a smooth submanifold $X \subset N$ so that any $q \in X$ is a regular value of f. Then f is a submersion at any $p \in f^{-1}(X)$, and thus the transversality condition (1) holds trivially. In particular,

Proposition 2.3. *If $f : M \to N$ is a submersion, then f intersects with any smooth submanifold X in N transversally.*

Now we can answer the question proposed above.

Theorem 2.4. *Let $f : M \to N$ be a smooth map, and let $X \subset N$ be a smooth submanifold so that $f \pitchfork X$. Then $f^{-1}(X)$ is a smooth submanifold in M whose codimension equals to the codimension of X (in N), and*

$$T_p(f^{-1}(X)) = df_p^{-1}(T_{f(p)}X), \qquad \forall p \in f^{-1}(X).$$

Proof. The argument above shows that if $f \pitchfork X$, then 0 is a regular value of $g \circ f$. So $f^{-1}(X) = (g \circ f)^{-1}(0)$ is a smooth submanifold in M. The dimension of $f^{-1}(X)$ is $\dim M - l$, where $l = \dim N - \dim X$. So $\dim M - \dim f^{-1}(X) = \dim N - \dim X$, i.e.

$$\mathrm{codim} f^{-1}(X) = \mathrm{codim} X.$$

Finally, by Theorem 1.1, the tangent space of $f^{-1}(X)$ at p is

$$T_p(f^{-1}(X)) = \ker(d(g\circ f)_p) = (dg_{f(p)}\circ df_p)^{-1}(0) = df_p^{-1}(dg_{f(p)}^{-1}(0)) = df_p^{-1}(T_{f(p)}X).$$

This completes the proof. $\square$

Of course if we take $X = \{p\}$, we will get Theorem 1.1.

Here is another interesting special case. Let X_1, X_2 be two smooth submanifolds of M and let $\iota : X_1 \hookrightarrow M$ be the canonical embedding. Then $\iota^{-1}(X_2) = X_1 \cap X_2$. Moreover, since we have the identification $T_pX_1 = d\iota_p(T_pX_1)$, the condition $\iota \pitchfork X_2$ is equivalent to say $T_pX_1 + T_pX_2 = T_pM$ for any $p \in X_1 \cap X_2$.

Definition 2.5. We say two smooth submanifolds X_1 and X_2 in M *intersect transversally* if for any $p \in X_1 \cap X_2$,

$$T_pX_1 + T_pX_2 = T_pM.$$

In this case we write $X_1 \pitchfork X_2$.

Remark. Note: By definition, if $X_1 \cap X_2 = \emptyset$, then $X_1 \pitchfork X_2$.

So if $X_1 \pitchfork X_2$, then $\iota \pitchfork X_2$. So $X_1 \cap X_2 = \iota^{-1}(X_2)$ is a smooth submanifold of M.Moreover, the dimension of $X_1 \cap X_2$ is given by

$$\dim(X_1\cap X_2) = \dim X_1 - (\dim M - \dim X_2) = \dim X_1 + \dim X_2 - \dim M.$$

Moreover, the tangent space of $X_1 \cap X_2$ at p equals $d\iota_p^{-1}(T_pX_2) = T_pX_1 \cap T_pX_2$. We conclude

Corollary 2.6. *Let X_1 and X_2 be two smooth submanifolds in M which intersect transversally, then $X_1 \cap X_2$ is a smooth submanifold whose dimension equals $\dim X_1 + \dim X_2 - \dim M$, and for any $p \in X_1 \cap X_2$,*

$$T_p(X_1 \cap X_2) = T_pX_1 \cap T_pX_2.$$

Remark. Note that if $X_1 \pitchfork X_2$, then the codimensions of X_1, X_2 and $X_1 \cap X_2$ in M satisfy the simple relation

$$\mathrm{codim} X_1 \cap X_2 = \mathrm{codim} X_1 + \mathrm{codim} X_2.$$

Finally we prove the following transversality theorem which is useful in finding transversal maps: (In general M could be a manifold with boundary, but we will not discuss that here.)

Theorem 2.7 (The Transversality Theorem)**.** *Let $F : S \times M \to N$ be a smooth map, $X \subset N$ be a smooth submanifold. For each $s \in S$, we let $f_s : M \to N$ be the map*

$$f_s(p) = F(s,p).$$

Suppose $F \pitchfork X$. Then for every regular value $s \in S$ of the projection map [1]

$$\pi : F^{-1}(X) \subset S \times M \to S, \quad \pi(s,p) = s,$$

one has $f_s \pitchfork X$. (So by Sard's theorem, for most $s \in S$, we have $f_s \pitchfork X$.)

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[1] Note that since $F \pitchfork X$, the pre-image $F^{-1}(X)$ is a smooth submanifold in $S \times M$. So the projection map π is a smooth map.

Proof. Let s be any regular value of π. For any $p \in f_s^{-1}(X)$, we need to show

$$\mathrm{Im}(df_s)_p + T_qX = T_qN,$$

where $q = f_s(p)$. Since $F \pitchfork X$, for any $Y_q \in T_qN$, there exists $(Z_s, Z_p) \in T_{(s,p)}(S \times M)$ and $Z_q \in T_qX$ such that

$$Y_q = (dF)_{(s,p)}(Z_s, Z_p) + Z_q.$$

But since s is a regular value of π, for $Z_s \in T_sS$, there exists $Z_p' \in T_pM$ so that $(Z_s, Z_p') \in T_{(s,p)}F^{-1}(X)$. So we can write

$$Y_q = (dF)_{(s,p)}(0, Z_p - Z_p') + (dF)_{(s,p)}(Z_s, Z_p') + Z_q.$$

The conclusion follows since

$$(dF)_{(s,p)}(0, Z_p - Z_p') = (df_s)_p(Z_p - Z_p') \in \mathrm{Im}(df_s)_p,$$

and

$$(dF)_{(s,p)}(Z_s, Z_p') \in dF_{(s,p)}\left(T_{(s,p)}F^{-1}(X)\right) \subset T_qX.$$

$\square$

As a corollary, we can show that transversal maps are generic:

Corollary 2.8. *Let $f : M \to \mathbb{R}^K$ be any smooth map and $X \subset \mathbb{R}^K$ be any smooth submanifold. Then for a.e. $v \in \mathbb{R}^K$, the "v-shifted" map*

$$f_v : M \to \mathbb{R}^K, \qquad p \mapsto f_v(p) = f(p) + v$$

is transversal to X.

Proof. Define a smooth map F by

$$F : M \times \mathbb{R}^K \to \mathbb{R}^K, \qquad (p, v) \mapsto f(p) + v.$$

Then for any fixed $p \in M$, $F(p, \cdot)$ is a submersion (a diffeomorphism in fact). It follows that F is a submersion from $M \times \mathbb{R}^K$ onto $\mathbb{R}^K$. So by Proposition 2.3, F intersects with any smooth submanifold X in $\mathbb{R}^K$ transversally. Now the conclusion follows from the transversality theorem above. $\square$

In particular, if we take f to be the inclusion map, we have

Corollary 2.9. *Let M, N be smooth submanifolds of $\mathbb{R}^K$. Then for a.e. $a \in \mathbb{R}^K$, $M + a$ intersects N transversally.*

(To get a better understanding of the previous two corollaries, think of the following problem: Let M and N be two spheres in $\mathbb{R}^K$. When will $M + a \pitchfork N$?)

By using tubular neighborhoods, one can easily prove

Corollary 2.10 (Transversality Homotopy Theorem)**.** *Let $f : M \to N$ be any smooth map, and $X \subset N$ be any smooth submanifold. Then f is homotopic to a smooth map $g : M \to N$ which intersects X transversally.*

Proof. Exercise. Hint: Embed N into $\mathbb{R}^K$. Let $\pi_\varepsilon : N^\varepsilon \to N$ be the ε-neighborhood of N. Now define $F : B \times M \to N$ as $F(s, p) = \pi_\varepsilon(f(p) + s\varepsilon(f(p)))$, where B is the open unit ball in $\mathbb{R}^K$. $\square$