

## LECTURE 12: SMOOTH VECTOR FIELDS

### 1. SMOOTH VECTOR FIELDS

Recall that a tangent vector  $X_p$  at  $p \in M$  is a linear map  $X_p : C^\infty(M) \rightarrow \mathbb{R}$  s.t.

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g).$$

We have known that we can replace  $C^\infty(M)$  by  $C^\infty(U)$ , where  $U$  is any open neighborhood of  $p$ . In particular, we can choose  $U$  to be a coordinate chart centered at  $p$ . In this case, we have a local diffeomorphism  $\varphi : U \rightarrow V \subset \mathbb{R}^n$ . Using the standard basis  $\left\{ \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \mid 1 \leq i \leq n \right\}$  of  $T_{\varphi(p)}\mathbb{R}^n$ ,<sup>1</sup> we can write down an explicit basis of  $T_pM$ :

$$(\partial_i)_p = (d\varphi^{-1})_{\varphi(p)} \left( \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right), \quad 1 \leq i \leq n.$$

When viewed as linear maps,  $(\partial_i)_p$  is given by

$$(\partial_i)_p : C^\infty(U) \rightarrow \mathbb{R}, \quad f \mapsto (\partial_i f)_p = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$$

If we embed  $M$  into some  $\mathbb{R}^K$ , then one can visualize  $(\partial_i)_p$  as the tangent vector (in the sense that we learned in calculus) of the curve (which is a curve in  $\mathbb{R}^K$ )

$$\gamma(t) = \varphi^{-1}((0, \dots, 0, t, 0, \dots, 0)), \quad t \in (-\varepsilon, \varepsilon)$$

at  $p$ , where  $t$  locates at the  $i^{\text{th}}$ -position,  $\varepsilon$  is chosen so that  $(0, \dots, 0, t, 0, \dots, 0) \in V$  for all  $-\varepsilon < t < \varepsilon$ , and we have chosen the coordinates so that  $\varphi(p) = 0 \in V \subset \mathbb{R}^n$ . By this way, for any vector  $X_p \in T_pM$ , we can write

$$X_p = \sum_{i=1}^n a_i (\partial_i)_p$$

and thus view  $X_p$  as a vector in  $\mathbb{R}^K$  (which is tangent to  $M$ ).

In calculus and in ODE, we have learned the conception of (smooth) vector fields, that is, an “smooth assignment”<sup>2</sup> of vectors to each point in the domain. In other words, a vector field  $X$  on  $V \subset \mathbb{R}^n$  assigns to each  $x \in V$  a vector  $X_x$ , and the smoothness assumption requires that if we write

$$X_x = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x^i} \Big|_x,$$

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<sup>1</sup>In other words, the identification  $T_{\varphi(p)}\mathbb{R}^n \simeq \mathbb{R}^n$  is given by  $\frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \leftrightarrow e_i$ , where  $\{e_1, \dots, e_n\}$  is the standard basis of  $\mathbb{R}^n$ .

<sup>2</sup>Of course one can also define continuous vector fields or even discontinuous vector fields. But unless otherwise stated, in this course we are only interested in smooth objects.

then the coefficients  $a_i(x)$  is smooth for  $x \in V$ . Here is the advantage of writing vector fields using “partial derivatives” instead of using Euclidian vector  $e_i$ 's:  $X$  is not merely a collection of arrows, but in fact a first order differential operator on  $C^\infty(V)$ . So vector field is an important object in analysis.

Now we define smooth vector fields on smooth manifolds. An obvious one is

**Definition 1.1.** A *smooth vector field*  $X$  on a smooth manifold  $M$  a smooth assignment that assigns to each  $p \in M$  a tangent vector  $X_p \in T_pM$ .

Of course one should explain the meaning of the words “smooth assignment”. This can be understood via local charts  $(\varphi, U, V)$ . In any such local chart, we note that  $\frac{\partial}{\partial x^i}$  are Euclidian smooth vector fields that are defined everywhere in  $V$ , and thus  $\partial_i$  should be smooth vector fields that are defined everywhere in  $U$ . For the vector field  $X$  under consideration, we can write

$$X = X^1\partial_1 + \cdots + X^n\partial_n = \sum X^i\partial_i,$$

where  $X^i$ 's are functions on  $U$ . We say  $X$  is smooth on  $U$ , if all coefficients  $X^i$ 's are smooth functions on  $U$ . (So again a smooth vector field  $X$  is a 1<sup>st</sup> order differential operator with smooth coefficients.)

One may worry that whether the above description depends on the choice of local charts. The following proposition gives us a global characterization of smooth vector fields, and can be used as an alternative (and more precise) definition.

Note that another way to say “assign to each  $p \in M$  a vector  $X_p \in T_pM$ ” is “give a map  $X : M \rightarrow TM$  so that  $\pi \circ X = \text{Id}$ ”, where  $\pi : TM \rightarrow M$  is the canonical projection  $\pi(X_p) = p$ , which, as we have seen in PSet 2 Problem 9, is a submersion from the smooth manifold  $TM$  (=the tangent bundle) to  $M$ .

**Proposition 1.2** (Alternative Definition of Smooth Vector Field). A *smooth vector field*  $X$  on  $M$  is a smooth map  $X : M \rightarrow TM$  such that  $\pi \circ X = \text{Id}$ .

*Proof.* We use local chart  $(\varphi, U, V)$  for  $M$ , and local chart  $(T\varphi, \pi^{-1}(U), V \times \mathbb{R}^n)$  for  $TM$ , where  $T\varphi$  is given by

$$T\varphi(p, X_p) = (\varphi(p), d\varphi_p(X_p)),$$

where when viewed as a vector in  $\mathbb{R}^n$ ,  $d\varphi_p(X_p)$  should be explained as the coefficients of  $d\varphi_p(X_p)$  in the basis  $\frac{\partial}{\partial x^i}|_{\varphi(p)}$ . Note that by definition  $d\varphi(\partial_i) = \frac{\partial}{\partial x^i}$ , which implies (by linearity)

$$X_p = \sum X^i(p)(\partial_i)_p \implies d\varphi_p(X_p) = \sum X^i(\varphi^{-1}(x)) \frac{\partial}{\partial x^i} \Big|_{\varphi(p)}.$$

So in these local charts, the map  $X$  becomes

$$T\varphi \circ X \circ \varphi^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^n, X^1(\varphi^{-1}(x)), \dots, X^n(\varphi^{-1}(x))).$$

By definition,  $X$  is a smooth map if and only if all these  $X^i$ 's are smooth.  $\square$

*Remark.* It is very important that although the vector field  $X$  is globally defined (=is defined everywhere AND is independent of local charts), the set of vector fields  $\partial_i$ 's and the coefficients  $X^i$ 's are only locally defined (=are defined only on  $U$  AND depend on the choice of the chart map). Although at each point, or even in each coordinate chart, one can write vector fields as linear combinations of  $n$  special vector fields, in general there do not exist exactly  $n$  global vector fields defined on  $M$  which "generate" all vector fields on  $M$ . ( $\rightarrow$  a strong topological restriction for  $M$ )

We will denote the set of all smooth vector fields on  $M$  by  $\Gamma^\infty(TM)$ . Note that  $\Gamma^\infty(TM)$  has a very nice algebraic structure:

$$X_1, X_2 \in \Gamma^\infty(TM), f_1, f_2 \in C^\infty(M) \implies f_1 X_1 + f_2 X_2 \in \Gamma^\infty(TM).$$

This implies that  $\Gamma^\infty(TM)$  is not only a vector space (whose dimension is infinity), but in fact a *module* over the algebra  $C^\infty(M)$ .

Now let  $X$  be any vector field (not necessary smooth) on  $M$ . Take any  $f \in C^\infty(M)$ , then for any  $p \in M$  we get a number  $X_p(f) \in \mathbb{R}$ . So  $X$  maps  $f$  to a function

$$Xf : M \rightarrow \mathbb{R}, \quad p \mapsto Xf(p) := X_p(f).$$

**Proposition 1.3.**  *$X$  is smooth if and only if*

$$f \in C^\infty(M) \implies Xf \in C^\infty(M).$$

*Proof.* Exercise. □

Let  $X$  be a smooth vector field on  $M$ . According to the Leibnitz rule for the tangent vector  $X_p : C^\infty(M) \rightarrow \mathbb{R}$ , we immediately get the Leibnitz rule for vector fields:

$$X(fg) = fX(g) + X(f)g, \quad \forall f, g \in C^\infty(M)$$

**Definition 1.4.** A *derivation* of the algebra  $C^\infty(M)$  is a map  $D : C^\infty(M) \rightarrow C^\infty(M)$  that satisfies the Leibnitz rule

$$D(fg) = fD(g) + D(f)g, \quad \forall f, g \in C^\infty(M).$$

So any smooth vector field  $X$  on  $M$  is a derivation. Conversely we have

**Proposition 1.5.** *Derivations are smooth vector fields: For any derivation  $D$  on  $C^\infty(M)$ , there is a smooth vector field  $X$ , such that  $Df = Xf$ .*

*Proof.* Let  $D$  be a derivation. For any  $p \in M$ , define  $X_p : C^\infty(M) \rightarrow \mathbb{R}$  by

$$X_p(f) := Df(p).$$

Then one can check that  $X_p$  is a derivative at  $p$ , in the sense of Lecture 5. As in Lecture 5,  $X_p$  must be a tangent vector at  $p$ . So starting from  $D$  we get a vector field  $X$  on  $M$ , with  $Df = Xf$ . The smoothness of  $X$  follows from Proposition 1.3. □

This can be thought of as a third (equivalent) definition of smooth vector field.

## 2. THE LIE BRACKET OF TWO VECTOR FIELDS

We learned in multivariable calculus: one of the most important properties of partial derivatives is that they commutes, i.e.  $\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i}$ . It turns out that the same property holds for the vector fields  $\partial_i$ 's, no matter what local chart you use:

**Lemma 2.1.** *For any chart  $(\varphi, U, V)$ , we have  $\partial_i \partial_j = \partial_j \partial_i$ .*

*Proof.* (This is not that obvious as it looks.) For any  $f \in C^\infty(U)$ , we have

$$\partial_i \partial_j f = \partial_i \left[ \frac{\partial(f \circ \varphi^{-1})}{\partial x^j} \circ \varphi \right] = \frac{\partial \left[ \frac{\partial(f \circ \varphi^{-1})}{\partial x^j} \circ \varphi \circ \varphi^{-1} \right]}{\partial x^i} \circ \varphi = \frac{\partial^2(f \circ \varphi^{-1})}{\partial x^i \partial x^j} \circ \varphi,$$

and the conclusion follows.  $\square$

Similarly, given  $X, Y \in \Gamma^\infty(TM)$ , we can define  $XY$  as the composition

$$XY : C^\infty(M) \rightarrow C^\infty(M), \quad f \mapsto X(Yf).$$

In general we no longer have  $XY = YX$ . However,

**Proposition 2.2.** *If  $X, Y \in \Gamma^\infty(TM)$ , then  $XY - YX \in \Gamma^\infty(TM)$ .*

*Proof.* There are several proofs. For example, one can check that  $X_p \circ Y - Y_p \circ X$  is a tangent vector at  $p$ , or check that  $XY - YX$  is a derivative. Here we give a proof via local coordinates. Take any chart  $(\varphi, U, V)$ . Denote  $X = \sum X^i \partial_i$  and  $Y = \sum Y^j \partial_j$ . Then for any  $f \in C^\infty(U)$ ,

$$X(Yf) - Y(Xf) = \sum_{i,j} X^i \partial_i (Y^j \partial_j f) - Y^i \partial_i (X^j \partial_j f) = \sum_{i,j} (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j f.$$

In other words, in the chosen local chart we have

$$XY - YX = \sum_{i,j} (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j.$$

So  $XY - YX \in \Gamma^\infty(TM)$ .  $\square$

*Remark.* As we have mentioned, vector fields can be regarded as first order differential operators. This phenomena above generalize to higher order differential operators: If  $P_1$  is a differential operator of order  $m_1$ ,  $P_2$  is a differential operator of order  $m_2$ , then  $P_1 \circ P_2$  is a differential operator of order  $m_1 + m_2$ , but  $P_1 \circ P_2 - P_2 \circ P_1$  is a differential operator of order (no more than)  $m_1 + m_2 - 1$ .

**Definition 2.3.** For any  $X, Y \in \Gamma^\infty(TM)$ , we call the commutator

$$[X, Y] = XY - YX$$

the *Lie bracket* of  $X$  and  $Y$ .

**WARNING:** It makes no sense to talk about things like  $X_p(Y_p f)$  or  $[X_p, Y_p]$ .