LECTURE 13: GEOMETRY AND DYNAMICS OF SMOOTH VECTOR FIELDS

1. GEOMETRY OF VECTOR FIELDS: THE INTEGRAL CURVES

Suppose we have a smooth vector field defined on an Euclidian region. In calculus and in ODE, we learned the conception of integral curves of such a vector field: they are curves so that the given vector field is the tangent vector to the curves everywhere.

Here is an example of vector fields with many integral curves drawn:

The conception of integral curves above can be generalized to smooth manifolds easily. To begin with, one need to explain the conceptions of “curve” and “tangent vector to a curve” first.

Suppose $M$ is a smooth manifold. A smooth curve in $M$ is by definition a smooth map $\gamma : I \to M$, where $I$ is an interval in $\mathbb{R}$. For any $a \in I$, the tangent vector of $\gamma$ at the point $\gamma(a)$ is defined to be

$$\dot{\gamma}(a) = \frac{d\gamma}{dt}(a) := d\gamma_a\left(\frac{d}{dt}\right),$$

where $\frac{d}{dt}$ is the standard coordinate tangent vector of $\mathbb{R}$.

**Definition 1.1.** Let $X \in \Gamma^\infty(TM)$ be a smooth vector field on $M$. A smooth curve $\gamma : I \to M$ is called an integral curve of $X$ if for any $t \in I$,

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

Note: By a curve we really mean a “parametrized curve”. The parametrization is a part of the definition. Different parametrizations of the “same geometric picture” represent different curves.
Example. Consider the coordinate vector field \( X = \frac{\partial}{\partial x^1} \) on \( \mathbb{R}^n \). Then the integral curves of \( X \) are the straight lines parallel to the \( x^1 \)-axis, parametrized as 
\[
\gamma(t) = (c_1 + t, c_2, \cdots, c_n).
\]
To check this, we note that for any smooth function \( f \) on \( \mathbb{R}^n \),
\[
d(\frac{d}{dt})f = \frac{d}{dt}(f \circ \gamma) = \nabla f \cdot \frac{d\gamma}{dt} = \frac{\partial f}{\partial x^1}.
\]
Remark. Note that although the curve \( \tilde{\gamma}(t) = (c_1 + 2t, c_2, \cdots, c_n) \) has the same picture (i.e. the same “horizontal line” passing the point \((c_1, \cdots, c_n)\)) as \( \gamma \), it is not an integral curve of \( X \), since \( \dot{\tilde{\gamma}}(t) = 2 \frac{\partial}{\partial x^1} \).

Example. Consider the vector field \( X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \) on \( \mathbb{R}^2 \). Then if \( \gamma(t) = (x(t), y(t)) \) is an integral curve of \( X \), we must have for any \( f \in C^\infty(\mathbb{R}^2) \),
\[
x'(t) \frac{\partial f}{\partial x} + y'(t) \frac{\partial f}{\partial y} = \nabla f \cdot \frac{d\gamma}{dt} = X_{\gamma(t)} f = x(t) \frac{\partial f}{\partial y} - y(t) \frac{\partial f}{\partial x},
\]
which is equivalent to the system
\[
x'(t) = -y(t), \quad y'(t) = x(t).
\]
The solution to this system is
\[
x(t) = a \cos t - b \sin t, \quad y(t) = a \sin t + b \cos t.
\]
These are circles centered at the origin in the plane parametrized by the angle (with counterclockwise orientation).

Remark. In general, a re-parametrization of an integral curve is no longer an integral curve. However, it is not hard to see that if \( \gamma : I \to M \) is an integral curve of \( X \), then
\[
\cdot \text{ Let } I_a = \{ t \mid t + a \in I \} \text{ and } \gamma_a(t) := \gamma(t + a), \text{ then } \gamma_a : I_a \to M \text{ is an integral curve of } X.
\]
\[
\cdot \text{ Let } I^a = \{ t \mid at \in I \} \text{ and } \gamma^a(t) := \gamma(at), \text{ then } \gamma^a : I^a \to M \text{ is an integral curve for } X^a = aX.
\]

To study further properties of integral curves, we need to convert the equation \( \dot{\gamma}(t) = X_{\gamma(t)} \) which is an equation relating tangent vectors on manifolds into equations on functions defined on Euclidian region. To do so we first prove

**Lemma 1.2.** Let \( X \) be a smooth vector field on \( M \), and suppose in a local chart \((\varphi, U, V)\), \( X = \sum X^i \partial_i \). Denote \( \varphi(p) = (x^1(p), \cdots, x^n(p)) \) (so that \( x^1, \cdots, x^n \) are smooth functions on \( U \) which represent the coordinates of \( p \)). Then \( X^i = X(x^i) \).

**Proof.** Since \( \partial_i(x^j) = \delta^j_i \) (Check This!), we have \( X(x^i) = \sum X^i \partial_i(x^j) = X^j \). \( \square \)
Now let $\gamma : I \to M$ be an integral curve of $X$. To study the equation $\dot{\gamma}(t) = X_{\gamma(t)}$ at a given point $\gamma(t)$, WLOG we may assume $\gamma(t) \in U$, and $(\varphi, U, V)$ is a coordinate chart. By using the local chart map $\varphi$, one can convert the point $\gamma(t) \in U$ to

$$\varphi(\gamma(t)) = (x^1(\gamma(t)), \ldots, x^n(\gamma(t))) \in \mathbb{R}^n.$$ 

If we denote $y_i = x^i \circ \gamma : I \to \mathbb{R}$, then we can convert the (vector!) equation defining integral curves into equations on these one-variable functions $y_i$’s. More precisely, according to the previous lemma, we have

$$\dot{\gamma}(t) = d\gamma\left(\frac{d}{dt}\right) = \sum_i d\gamma_i\left(\frac{d}{dt}\right)(x^i)\partial_i = \sum_i (x^i \circ \gamma)'(t)\partial_i = \sum_i y'_i(t)\partial_i.$$ 

So the integral curve equation $\dot{\gamma}(t) = X_{\gamma(t)}$ becomes

$$\sum_i y'_i(t)\partial_i = \sum_i X^i(\gamma(t))\partial_i = \sum_i X^i \circ \varphi^{-1}(y_1(t), \ldots, y_n(t)).$$

for all $t \in I$. In conclusion, we convert the integral curve equation into the following system of ODEs on the one-variable functions $y_i$’s:

$$y'_i(t) = X^i \circ \varphi^{-1}(y_1, \ldots, y_n), \quad \forall t \in I, \forall 1 \leq i \leq n.$$ 

Recall:

**Theorem 1.3** (The Fundamental Theorem for Systems of First Order ODEs). Suppose $V \subset \mathbb{R}^n$ is open, and $F = (F^1, \ldots, F^n) : V \to \mathbb{R}^n$ a smooth vector-valued function. Consider the initial value problem

$$\begin{cases}
\dot{y}^i(t) = F^i(y^1(t), \ldots, y^n(t)), & i = 1, \ldots, n \\
y^i(t_0) = c^i, & i = 1, \ldots, n
\end{cases}$$

for arbitrary $t_0 \in \mathbb{R}$ and $c_0 = (c^1, \ldots, c^n) \in V$.

1. **Existence**: For any $t_0 \in \mathbb{R}$ and any $c_0 \in V$, there exist an open interval $I_0$ containing $t_0$ and an open subset $V_0$ containing $c_0$ so that for any $c \in V_0$, the system (1) has a smooth solution $y_c(t) = (y^1(t), \ldots, y^n(t))$ for $t \in I_0$.
2. **Uniqueness**: If $y_1$ is a solution to the system (1) for $t \in I_0$ and $y_2$ is a solution to the system (1) for $t \in J_0$, then $y_1 = y_2$ for $t \in I_0 \cap J_0$.
3. **Smoothness**: The solution function $Y(c, t) := y_c(t)$ in part (1) is smooth on $(c, t) \in V_0 \times I_0$.

We will refer to Lee’s book, Appendix D (Page 663-671) for a proof. According to the fundamental theorem of systems of ODEs, we conclude

**Theorem 1.4** (Local Existence, Uniqueness and Smoothness). Suppose $X$ is a smooth vector field on $M$. Then for any point $p \in M$, there exists a neighborhood $U_p$ of $p$, an $\varepsilon_p > 0$ and a smooth map

$$\Gamma : (-\varepsilon_p, \varepsilon_p) \times U_p \to M$$

so that for any $q \in U$, the curve $\gamma_q : (-\varepsilon, \varepsilon) \to M$ defined by

$$\gamma_q(t) := \Gamma(t, q)$$
is an integral curve of $X$ with $\gamma(0) = q$. Moreover, this integral curve is unique in the sense that if $\sigma : I \rightarrow M$ is another integral curve of $X$ with $\sigma(0) = q$, then $\sigma(t) = \gamma_q(t)$ for $t \in I \cap (-\varepsilon, \varepsilon)$.

2. Complete vector fields

As a consequence of the uniqueness, any integral curve has a maximal defining interval. We are interested in those vector fields whose maximal defining interval is $\mathbb{R}$.

**Definition 2.1.** A vector field $X$ on $M$ is complete if for any $p \in M$, there is an integral curve $\gamma : \mathbb{R} \rightarrow M$ such that $\gamma(0) = p$.

Note every smooth vector field is complete.

**Example.** Consider the vector field $X = t^2 \frac{d}{dt}$ on $\mathbb{R}$. Let $\gamma(t) = (x(t))$ be its integral curve. Then according to the integral curve equation,

$$x'(t) \frac{d}{dt} = X_{\gamma(t)} = x(t)^2 \frac{d}{dt} \implies x'(t) = x(t)^2.$$

The solution to this ODE is with initial condition $x(0) = c$ is

$$x_c(t) = \frac{1}{-t + 1/c} \quad \text{for } c \neq 0$$

and

$$x_0(t) = 0 \quad \text{for } c = 0.$$  

Note that the maximal interval of $x_c(t)$ is

$$I_c = (-\infty, 1/c) \quad \text{for } c > 0$$

and

$$I_c = (1/c, +\infty) \quad \text{for } c < 0.$$  

Since the integral curves starting at any $c \neq 0$ is not defined for all $t \in \mathbb{R}$, we conclude that $X$ is not complete.

We will use complete vector fields to construct global flows next time. We will end this lecture with a sufficient condition for a vector field to be complete. As in the case of functions, we can define the support of a vector field by

$$\text{supp}(X) = \{p \in M \mid X(p) \neq 0\}.$$  

Our criteria is

**Theorem 2.2.** If $X$ is a compactly supported vector field on $M$, then it is complete.

**Proof.** Let $C = \text{supp}(X)$. First suppose $q \in M \setminus C$, i.e. $X_q = 0$. We define a “constant curve” $\gamma_q$ on $M$ by letting $\gamma_q(t) = q$ for all $t \in \mathbb{R}$, then we see

$$\dot{\gamma}_q(t) = 0 = X_q = X_{\gamma_q(t)}.$$  

In other words, the constant curve $\gamma_q$ (whose domain is $\mathbb{R}$) is the unique integral curve of $X$ passing $q$. 

Now suppose $p \in C$. (The idea: use compactness to find a uniform constant $\varepsilon_0$ so that any integral curve starting at a point in $C$ is defined on an interval of length $2\varepsilon_0$. Then if we have an integral curve, we can always extend the domain by $\varepsilon_0$.) Since any integral curve starting at $q \in M \setminus C$ stays at $q$, we see that every integral curve starting at $p \in C$ stays in $C$. By Theorem 1.4, for any $q \in C$, there is an interval $I_q = (-\varepsilon_q, \varepsilon_q)$, a neighborhood $U_q$ of $q$ and a smooth map
\[ \Gamma : I_q \times U_q \to C \]
such that for all $p \in U_q$,
\[ \gamma_p(t) = \Gamma(t, p) \]
is an integral curve of $X$ with $\gamma_p(0) = p$. Since $\cup_q U_q \supset C$, and $C$ is compact, one can find a finite many points $q_1, \ldots, q_N$ in $C$ so that $\{U_{q_1}, \ldots, U_{q_N}\}$ cover $C$. Let $I = \cap_k I_{q_k} = (-\varepsilon_0, \varepsilon_0)$, then for any $q \in C$, there is an integral curve $\gamma_q : I \to C$. Now suppose the maximal defining interval for $p \in C$ is $I_p$. We need to prove $I_p = \mathbb{R}$. In fact, if $I_p \neq \mathbb{R}$, WLOG, we may assume that $\sup I_p = c < \infty$. Then starting from the point $q = \gamma_p(c - \frac{\varepsilon_0}{2})$, there is an integral curve
\[ \gamma_q : (-\varepsilon_0, \varepsilon_0) \to M \]
of the vector field $X$. By uniqueness, $\gamma_q(t) = \gamma_p(t + c - \frac{\varepsilon_0}{2})$. It follows that the defining interval of $\gamma_p$ extends to $c + \frac{\varepsilon_0}{2}$, which is a contradiction. \hfill \square

In particular, if $M$ itself is compact, then the set $\text{Supp}(X)$, as a closed set in the compact manifold, is always compact. So we get

**Corollary 2.3.** Any smooth vector field on a compact manifold is complete.