LECTURE 14: FLOWS

1. Flows

Now suppose M is a smooth manifold and X is a complete vector field on M. By definition, for any $p \in M$, there is a unique integral curve $\gamma_p : \mathbb{R} \to M$ such that $\gamma_p(0) = p$. From this one can, for any $t \in \mathbb{R}$, define a map

$$\phi_t : M \to M, \quad p \mapsto \gamma_p(t).$$

By definition, $\phi_t(p)$ is smooth in t for any fixed p. In fact, the family of maps $\{\phi_t \mid t \in \mathbb{R}\}$ satisfies the following very important *group law*:

Proposition 1.1. *For any $t, s \in \mathbb{R}$, we have $\phi_t \circ \phi_s = \phi_{t+s}$.*

Proof. Notice that for any $p \in M$ and any fixed $s \in \mathbb{R}$, the two curves

$$\gamma_1(t) = \phi_t \circ \phi_s(p) \quad \text{and} \quad \gamma_2(t) = \phi_{t+s}(p)$$

are both integral curves for X starting at the same point

$$\gamma_1(0) = \phi_s(p) = \gamma_2(0).$$

By uniqueness of integral curves, we have

$$\phi_t \circ \phi_s = \phi_{t+s}.$$

□

Since $\phi_0 = \mathrm{Id}$, we conclude that

Corollary 1.2. *$\phi_t : M \to M$ is bijective, and $\phi_t^{-1} = \phi_{-t}$.*

In what follows we will prove that each ϕ_t, as a map from M to M, is smooth, and thus is a diffeomorphism. As a consequence, the map

$$t \mapsto \phi_t$$

is a group homomorphism from $\mathbb{R}$ to $\mathrm{Diff}(M)$. We will call the family $\{\phi_t \mid t \in \mathbb{R}\}$ a *one-parameter subgroup of diffeomorphisms*.

In fact, the result we are going to prove below is a bit stronger: if we let Φ be the collection of all ϕ_t's, then Φ is smooth as a map on joint variables (t, p). It follows that each ϕ_t is smooth, since it can be written as a composition of smooth maps

$$\begin{array}{ccccc} M & \longrightarrow & \mathbb{R} \times M & \xrightarrow{\Phi} & M \\ p & \mapsto & (t,p) & \mapsto & \Phi(t,p) \end{array}$$

Theorem 1.3. *The map*

$$\Phi : \mathbb{R} \times M \to M, \quad (t,p) \mapsto \phi_t(p)$$

is smooth.

Proof. According to the fundamental theorem, for any $p \in M$, there is a neighborhood U_p of p and an interval $I_p = (-\varepsilon_p, \varepsilon_p) \ni 0$ such that $\Phi|_{I_p \times U_p} : I_p \times U_p \to M$ is smooth. We need to prove that Φ is smooth near a point $(t_0, p) \in \mathbb{R} \times M$ for larger t_0.

Here is the idea: By group law we know $\gamma_p(t_0 + t) = \gamma_{\gamma_p(t_0)}(t)$. Since "joint smoothness" holds for small t, we can try to connect p to $\gamma_{t_0}(p)$ by a sequence of points, so that each point can be joint to the next point by a "short" integral curve on which the smoothness holds uniformly for small t. To get such a finite sequence of points, we apply our favorite compactness argument.

Since $\gamma_p : \mathbb{R} \to M$ is smooth, the set $K = \gamma_p([-\varepsilon_p, t_0 + \varepsilon_p])$ is compact. So one can find finitely many points $p_1, \cdots, p_m$ in K so that the open sets $U_{p_1}, \cdots, U_{p_m}$ cover K. As last time, we denote

$$U := \bigcup_k U_{p_k} \qquad \text{and} \qquad I = (-\varepsilon_0, \varepsilon_0) := \bigcap_k I_{p_k}.$$

Then U is an open neighborhood of K, I is an open interval containing 0, and the map

$$\Phi|_{I \times U} : I \times U \to M$$

is smooth. In particular, for any $s_0 \in I$, the map $\phi_{s_0} = \Phi(s_0, \cdot) : U \to M$ is smooth.

Now we take N large enough so that $s_0 := t_0/N \in I$. We set $U_1 = U$ and define iteratively

$$U_{k+1} := \phi_{s_0}^{-1}(U_k) \cap U, \qquad k = 1, \cdots, N-1.$$

Then for any $k = 1, \cdots, N$, U_k is open, and $\phi_{s_0} : U_k \to U_{k-1}$ is smooth. Finally we consider the open set $\Phi|_{I \times U}^{-1}(U_N)$. Since

$$\Phi(s_0, \Phi(s_0, \cdots, \Phi(s_0, \Phi(0, p)) \cdots)) = \Phi(t_0, p) = \gamma_p(t_0) \in K \subset U,$$

we see $(0, p) \in \Phi|_{I \times U}^{-1}(U_N)$. So there exists a neighborhood $I_0 \times U_0$ of $(0, p)$ so that $I_0 \times U_0 \subset \Phi|_{I \times U}^{-1}(U_N)$, i.e. $\Phi|_{I_0 \times U_0} : I_0 \times U_0 \to U_N$ is smooth. It follows that Φ is smooth on $(t_0 + I_0) \times U_0$, since it can be written as a composition

$$\begin{array}{ccccccc} (t_0 + I_0) \times U_0 & \longrightarrow I_0 \times U_0 & \xrightarrow{\Phi} U_N & \xrightarrow{\phi_{s_0}} & \cdots & \xrightarrow{\phi_{s_0}} U_1 & \xrightarrow{\phi_{s_0}} U_0 \\ (t, p) & \mapsto (t - t_0, p) & \mapsto & & \cdots & & \mapsto \Phi(t, p) \end{array}$$

of smooth maps. This completes the proof. □

Notice that the group law can also be rewritten in terms of the map Φ as

$$\Phi(t + s, p) = \Phi(t, \Phi(s, p)).$$

Definition 1.4. We will call $\Phi : \mathbb{R} \times M \to M$, $(t, p) \mapsto \phi_t(p)$ the *flow* of X.

Remark. In the definition above we assume that X is complete. If X is not complete, one can still develop a similar theory of *local flow* generated by X. More precisely, there exists an open subset $\mathcal{M}$ of the form

$$\{0\} \times M \subset \mathcal{M} = \{(t, p) \mid -t_p < t < t_p\} \subset \mathbb{R} \times M$$

and a smooth map

$$\Phi : \mathcal{M} \to M$$

such that if we write $\phi_t(p) = \Phi(t, p)$, then

1. $\phi_0 = \mathrm{Id}$.
2. $\phi_{t+s} = \phi_t \circ \phi_s$ if $t, s, t+s \in (-t_p, t_p)$.

Moreover, if we write $\gamma_p(t) = \Phi(t, p)$ for $t \in (-t_p, t_p)$, then γ_p is the unique integral curve of X passing p.

Example. The flow generated by the vector field $X = \frac{\partial}{\partial x^1}$ is the translation

$$\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, x^1, x^2, \cdots, x^n) \mapsto (t + x^1, x^2, \cdots, x^n).$$

More generally, the flow generated by a constant vector field $X = \sum c^i \frac{\partial}{\partial x^i}$ is

$$\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, x^1, x^2, \cdots, x^n) \mapsto (c^1 t + x^1, \cdots, c^n t + x^n).$$

Example. If we identify $\mathbb{R}^2$ with $\mathbb{C}$, then the flow generated by the vector field

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is the counterclockwise rotation

$$\Phi : \mathbb{R} \times \mathbb{C} \to \mathbb{C}, \quad (t, z) \mapsto e^{it} z.$$

Note that this vector field is tangent to circles centered at the origin. We will denote the induced vector fields on such circles by $\frac{d}{d\theta}$.

2. Lie Derivatives

A smooth manifold M together with a flow Φ is called a *dynamical system*. In what follows we study the dynamical system generated by the flow of a vector field X.

For simplicity we assume X is complete, so that we have a family of diffeomorphisms $\phi_t : M \to M$. Now let $f \in C^\infty(M)$ be any smooth function. (In physics, M represents the phase space, and f represents an observable.) We would like to calculate "the rate of change of f" along the flow.

Proposition 2.1. *Let $X \in \Gamma^\infty(TM)$ be complete. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \phi_t^* f = X(f), \qquad \forall f \in C^\infty(M).$$

Proof. Let $\gamma_p(t)$ be the integral curve of X with $\gamma_p(0) = p$, then

$$\phi_t^* f(p) = f(\phi_t(p)) = f(\gamma_p(t)).$$

Thus

$$\left.\frac{d}{dt}\right|_{t=0} \phi_t^* f(p) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma_p(t)) = d(f\circ\gamma_p)(\left.\frac{d}{dt}\right|_{t=0}) = df_p\circ(d\gamma_p)_0(\left.\frac{d}{dt}\right|_{t=0}) = df_p(X_p) = Xf(p).$$

□

Remark. The proposition holds for any smooth vector field X. To see this, one just need to notice that for any $p \in M$, there is a neighborhood U of p and a uniform t_0 so that $\phi_t(q) = \gamma_q(t)$ makes sense for all $q \in U$ and all $|t| < t_0$. Then repeat the proof above.

Definition 2.2. The *Lie derivative* of a $f \in C^\infty(M)$ with respect to $X \in \Gamma^\infty(TM)$ is

$$\mathcal{L}_X(f) := \frac{d}{dt}\Big|_{t=0} \phi_t^* f \quad \left(= \lim_{t \to 0} \frac{\phi_t^* f - f}{t}\right).$$

So the Lie derivative $\mathcal{L}_X f$ measures "the rate of change of a smooth function f along the direction of X". Now let $Y \in \Gamma^\infty(TM)$ be another smooth vector field. We would like to measure "the rate of change of a Y along the direction of X". One way to do so is to embed M into some Euclidian space and measure the rate of change of "coordinates components" of Y. In what follows we will take an intrinsic way. Naively one would like to compute the limit "$\lim_{t\to 0} \frac{Y_{\phi_t(p)} - Y_p}{t}$", where ϕ_t is the flow generated by X. Unfortunately $Y_{\phi(p)}$ is not a tangent vector at p and thus the expression "$Y_{\phi_t(p)} - Y_p$" makes no sense at all. To fix the problem, we need to "push-forward" the tangent vector $Y_{\phi_t(p)}$ at $\phi_t(p)$ to a tangent vector at p. This can be done via the so-called "push-forward" operation for vector fields, which is an analogy of the "pull-back" operation on smooth functions.

In general, given a smooth vector field $X \in \Gamma^\infty(TM)$, one cannot "push-forward" X to get a smooth vector field on N: for any $p \in M$, we know $d\varphi_p(X_p) \in T_{\varphi(p)}N$. However, "$Y_{\varphi(p)} := d\varphi_p(X_p)$" does not define a smooth vector field on N, since

- There may exists $q \in N$ which is not in the image of φ.
- There may exists $p_1, p_2 \in M$ such that $\varphi(p_1) = \varphi(p_2)$ but $d\varphi_{p_1}(X_{p_1}) \neq d\varphi_{p_2}(X_{p_2})$.

However, if $\varphi : M \to N$ is a diffeomorphism, then the two issues above do not exist and thus the formula $Y_{\varphi(p)} := d\varphi_p(X_p)$ does define a smooth vector field on N.

Definition 2.3. If $\varphi : M \to N$ is a diffeomorphism, and $X \in \Gamma^\infty(TM)$, then the *push-forward* of X is the smooth vector field $\varphi_* X$ on N defined by

$$(\varphi_* X)_{\varphi(p)} = d\varphi_p(X_p), \qquad \forall p \in M.$$

More generally, we define

Definition 2.4. Suppose $\varphi : M \to N$ is a smooth map. We say that $X \in \Gamma^\infty(TM)$ and $Y \in \Gamma^\infty(TN)$ are *φ-related* if

$$d\varphi_p(X_p) = Y_{\varphi(p)}, \qquad \forall p \in M.$$

Lemma 2.5. *Suppose $\varphi : M \to N$ is smooth and $X \in \Gamma^\infty(TM), Y \in \Gamma^\infty(TN)$ are φ-related. Then for any $g \in C^\infty(N)$, $X(\varphi^* g) = \varphi^*(Yg)$.*

Proof. Suppose $q = \varphi(p)$, then

$$\varphi^*(Yg)(p) = (Yg)(q) = Y_q g = d\varphi_p(X_p) g = X_p(g \circ \varphi) = X_p(\varphi^* g) = X(\varphi^* g)(p).$$

□

By definition, if $\varphi : M \to N$ is a diffeomorphism, then any vector field X is φ-related to its push-forward $\varphi_* X$. According to Lemma 2.5, we immediately get

Corollary 2.6. *If $\varphi : M \to N$ is a diffeomorphism, $(\varphi_* X) g = (\varphi^{-1})^* X \varphi^* g$.*

Now we define the "rate of change of $Y \in \Gamma^\infty(TM)$ with respect to $X \in \Gamma^\infty(TM)$.

Definition 2.7. The *Lie derivative* of $Y \in \Gamma^\infty(TM)$ with respect to $X \in \Gamma^\infty(TM)$ is

$$\mathcal{L}_X(Y) := \left.\frac{d}{dt}\right|_{t=0} (\phi_{-t})_* Y \quad \left(= \lim_{t \to 0} \frac{(\phi_{-t})_* Y_{\phi_t(p)} - Y_p}{t} \right).$$

Theorem 2.8. *Let $X, Y \in \Gamma^\infty(TM)$ and suppose X is complete. Then*

$$\mathcal{L}_X(Y) = [X, Y].$$

Proof. Since each ϕ_t is a diffeomorphism,

$$(\phi_{-t})_* Y = \phi_t^* Y \phi_{-t}^*.$$

So for any $f \in C^\infty(M)$,

$$\left.\frac{d}{dt}\right|_{t=0} (\phi_{-t})_* Y f = \left.\frac{d}{dt}\right|_{t=0} \phi_t^* Y \phi_{-t}^* f = \left.\frac{d}{dt}\right|_{t=0} \phi_t^* Y f + \left.\frac{d}{dt}\right|_{t=0} Y \phi_{-t}^* f = XYf - YXf.$$

□

Remark. Again one can remove the completeness assumption on X, since the differentiation above is local.

So roughly speaking, Xf can be think of as "the derivative of f along the direction of X", and $[X, Y]$ can be think of as "the derivative of Y along the direction of X". Later on we will define Lie derivatives for differential forms.