LECTURE 16: LIE GROUPS AND THEIR LIE ALGEBRAS

1. Lie groups

A Lie group is a special smooth manifold on which there is a *group* structure, and moreover, the two structures are compatible. Lie groups are not just special examples of smooth manifolds with nice properties. They represent smooth families of symmetries and play very important roles in various places in geometry and physics.

Definition 1.1. A *Lie group* G is a smooth manifold equipped with a group structure so that the group multiplication

$$\mu : G \times G \to G, \quad (g_1, g_2) \mapsto g_1 \cdot g_2$$

is a smooth map.

Example. Here are some basic examples:

- $\mathbb{R}^n$, considered as a group under addition.
- $\mathbb{R}^* = \mathbb{R} - \{0\}$, considered as a group under multiplication.
- S^1, considered as a group under multiplication.
- Linear Lie groups with matrix multiplications

$$\begin{aligned} \mathrm{GL}(n,\mathbb{R}) &= \{X \in M(n,\mathbb{R}) \mid \det X \neq 0\}, \quad \text{(the general linear group)} \\ \mathrm{SL}(n,\mathbb{R}) &= \{X \in M(n,\mathbb{R}) \mid \det X = 1\}, \quad \text{(the special linear group)} \\ \mathrm{O}(n) &= \{X \in M(n,\mathbb{R}) \mid XX^T = \mathrm{I}_n\}, \quad \text{(the orthogonal group).} \end{aligned}$$

  ($\mathrm{GL}(n,\mathbb{R})$ is a smooth manifold since it is an open set in $M(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$, $\mathrm{SL}(n,\mathbb{R})$ is a smooth manifold according to PSet 3 Part 2 Problem 4, $\mathrm{O}(n)$ is a smooth manifold by PSet 2 Problem 10.)
- If G_1 and G_2 are Lie groups, so is their product $G_1 \times G_2$. (So $\mathbb{T}^n = S^1 \times \cdots \times S^1$ is a Lie group.)

Remarks. (1) (Hilbert's 5^{th} problem, [Gleason and Montgomery-Zippin, 1950's]): Let G be any topological group whose underlying space is a topological manifold, then G admits a smooth structure to make it a Lie group.

(2) Not every smooth manifold admits a Lie group structure. For example, the only spheres that admit a Lie group structure are S^0, S^1 and S^3; (Can you write down a Lie group structure on S^3?) among all the compact 2 dimensional surfaces the only one that admits a Lie group structure is $T^2 = S^1 \times S^1$. A simple topological obstruction of the existence of Lie group structure is that $\pi_1(G)$ is always abelian. (Left as an exercise.)

Now suppose G is a Lie group. For any elements $a, b \in G$, there are two natural maps, the left multiplication

$$L_a : G \to G, \quad g \mapsto a \cdot g$$

and the right multiplication

$$R_b : G \to G, \quad g \mapsto g \cdot b.$$

L_a is smooth since it can be viewed as a composition of smooth maps,

$$\begin{aligned} L_a : G &\xhookrightarrow{j_a} G \times G \xrightarrow{\mu} G, \\ g &\longmapsto (a, g) \quad \longmapsto a \cdot g. \end{aligned}$$

Similarly $R_b = \mu \circ i_b$ is smooth, where $i_b : G \hookrightarrow G \times G$ is the smooth inclusion map $i_b(g) = (g, b)$. It is obviously that $L_a^{-1} = L_{a^{-1}}$ and $R_b^{-1} = R_{b^{-1}}$. So both L_a and R_b are diffeomorphisms. Moreover, L_a and R_b commutes with each other: $L_a R_b = R_b L_a$.

Lemma 1.2. *The differential of the multiplication map $\mu : G \times G \to G$ is given by*

$$d\mu_{a,b}(X_a, Y_b) = (dR_b)_a(X_a) + (dL_a)_b(Y_b)$$

for any $(X_a, Y_b) \in T_aG \times T_bG \simeq T_{(a,b)}(G \times G)$.

Proof. For any function $f \in C^\infty(G)$, we have

$$\begin{aligned} (d\mu_{a,b}(X_a, Y_b))(f) &= (X_a, Y_b)(f \circ \mu) \\ &= X_a(f \circ \mu \circ i_b) + Y_b(f \circ \mu \circ j_a) \\ &= X_a(f \circ R_b) + Y_b(f \circ L_a) \\ &= (dR_b)_a(X_a)(f) + (dL_a)_b(Y_b)(f). \end{aligned}$$

□

As an application, we can prove

Proposition 1.3. *For any Lie group G, the group inversion map*

$$i : G \to G, \quad g \mapsto g^{-1}$$

is smooth.

Proof. [Idea: write i as a composition of smooth maps...] Consider the smooth map

$$f : G \times G \to G \times G, \quad (a, b) \mapsto (a, ab).$$

Obviously it is bijective. According to the lemma above, the derivative of f is

$$df_{(a,b)} : T_aG \times T_bG \to T_aG \times T_{ab}G, \quad (X_a, Y_b) \mapsto (X_a, (dR_b)_a(X_a) + (dL_a)_b(Y_b)).$$

This is a bijective linear map since dR_b, dL_a are. (Can you write down the inverse of $df_{(a,b)}$?) It follows by the inverse function theorem that f is locally a diffeomorphism near each pair (a, b). However, since f is globally bijective, it must be a globally diffeomorphism. We conclude that its inverse,

$$f^{-1} : G \times G \to G \times G, \quad (a, c) \mapsto (a, a^{-1}c)$$

is a diffeomorphism. Thus the inversion map i, as the composition

$$\begin{array}{ccccccc} G & \hookrightarrow & G\times G & \xrightarrow{f^{-1}} & G\times G & \xrightarrow{\pi_2} & G \\ a & \longmapsto & (a,e) & \longmapsto & (a,a^{-1}) & \longmapsto & a^{-1} \end{array}$$

is smooth. □

(Can you compute $di_a(X_a)$?)

2. Lie algebras associated to Lie groups

We start with the abstract definition.

Definition 2.1. A *Lie algebra* is a real vector space V together with a binary bracket operation

$$[\cdot,\cdot] : V \times V \to V,$$

such that for any $X, Y, Z \in V$,

(1) (Linearity) $[aX_1 + bX_2, Y] = a[X_1, Y] + b[X_2, Y]$,
(2) (Skew-symmetry) $[X, Y] = -[Y, X]$,
(3) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The bracket $[\cdot,\cdot]$ is called the *Lie bracket*.

Example. Any vector space admits a trivial Lie algebra structure: $[X, Y] \equiv 0$.

Example. The set of all smooth vector fields $\Gamma^\infty(TM)$ on any smooth manifold form a Lie algebra. (See PSet 4 Part 1 Problem 2).

Example. The set of all $n \times n$ real matrices, $M(n, \mathbb{R})$, is a Lie algebra if we define the Lie bracket to be the commutator of matrices: $[A, B] = AB - BA$. (More generally, the commutator $[X, Y] = XY - YX$ define a Lie algebra structure on any associative algebra.

Now we will associate to any Lie group G a God-given Lie algebra. Suppose G is a Lie group. From the left translation L_a one can, for any vector $X_e \in T_eG$, define a vector field X on G by

$$X_a = (dL_a)_e(X_e).$$

One can use local coordinates to prove the smoothness of X.

It is not surprising that the vector field X is "invariant" under any left translation:

$$(dL_a)_b(X_b) = (dL_a)_b \circ dL_b(X_e) = dL_{ab}(X_e) = X_{ab}.$$

Definition 2.2. A *left invariant vector field* on a Lie group G is a smooth vector field X on G which satisfies $(dL_a)_b(X_b) = X_{ab}$.

So any tangent vector $X_e \in T_eG$ determines a left invariant vector field on G. Conversely, any left invariant vector field X is uniquely determined by its "value" X_e at $e \in G$, since for any $a \in G$, $X(a) = (dL_a)X_e$.

We will denote the set of all left invariant vector fields on Lie group G by $\mathfrak{g}$, i.e.

$$\mathfrak{g} = \{X \in \Gamma^\infty(TG) \mid X \text{ is left invariant}\}.$$

Then it is not hard to see that

- $\mathfrak{g}$ is a vector subspace of $\Gamma^\infty(TG)$.
- the correspondence $X_e \in T_eG \leftrightsquigarrow X \in \mathfrak{g}$ preserves the linear structures.

So we have a linear isomorphism

$$\mathfrak{g} \simeq T_eG$$

as vector spaces. In particular, $\dim \mathfrak{g} = \dim G$.

Next we show that $\mathfrak{g}$ is a Lie algebra, the Lie algebra structure being the one induced from the Lie algebra structure on $\Gamma^\infty(TG)$:

Proposition 2.3. *If $X, Y \in \mathfrak{g}$, so is their Lie bracket $[X, Y]$.*

Proof. We want to show that $[X, Y]$ is left-invariant if X and Y are. First notice

$$Y(f \circ L_a)(b) = Y_b(f \circ L_a) = (dL_a)_b(Y_b)f = Y_{ab}f = (Yf)(L_ab) = (Yf) \circ L_a(b)$$

for any smooth function $f \in C^\infty(G)$. Thus

$$X_{ab}(Yf) = (dL_a)_b(X_b)(Yf) = X_b((Yf) \circ L_a) = X_bY(f \circ L_a).$$

Similarly $Y_{ab}Xf = Y_bX(f \circ L_a)$. Thus

$$dL_a([X, Y]_b)f = X_bY(f \circ L_a) - Y_bX(f \circ L_a) = X_{ab}(Yf) - Y_{ab}(Xf) = [X, Y]_{ab}(f).$$

□

It follows that the space $\mathfrak{g}$ of all left invariant vector fields on G together with the Lie bracket operation $[\cdot, \cdot]$ is an n-dimensional *Lie subalgebra* of the Lie algebra of all smooth vector fields $\Gamma^\infty(TG)$.

Definition 2.4. The Lie algebra $\mathfrak{g}$ of is called *the Lie algebra of* G.

We give a couple examples:

Example. (The Euclidean space $\mathbb{R}^n$.) This is obviously a Lie group, since the group operation

$$\mu((x_1, \cdots, x_n), (y_1, \cdots, y_n)) := (x_1 + y_1, \cdots, x_n + y_n).$$

is smooth.

Moreover, for any $a \in \mathbb{R}^n$, the left translation L_a is just the usual translation map on $\mathbb{R}^n$. So dL_a *is* the identity map, as long as we identify $T_a\mathbb{R}^n$ with $\mathbb{R}^n$ in the usual way. It follows that any left invariant vector field is in fact a constant vector field, i.e.

$$X_v = v_1\frac{\partial}{\partial x_1} + \cdots + v_n\frac{\partial}{\partial x_n}$$

for $\vec{v} = (v_1, \cdots, v_n) \in T_0\mathbb{R}^n$. Since $\frac{\partial}{\partial x_i}$ commutes with $\frac{\partial}{\partial x_j}$ for any pair (i, j), we conclude that the Lie bracket of any two left invariant vector fields vanishes. In other words, the Lie algebra of $G = \mathbb{R}^n$ is $\mathfrak{g} = \mathbb{R}^n$ with vanishing Lie bracket.

Example. (The *general linear group*)

$$\mathrm{GL}(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) \mid \det X \neq 0\}.$$

Obviously $\mathrm{GL}(n,\mathbb{R})$ is n^2-dimensional noncompact Lie group. Moreover, it is not connected, but consists of exactly two connected components,

$$\mathrm{GL}_+(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) \mid \det X > 0\}$$

and

$$\mathrm{GL}_-(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) \mid \det X < 0\}.$$

The fact that $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $M(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$ implies that the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$, as the tangent space at $e = I_n$, is the set $M(n,\mathbb{R})$ itself, i.e.

$$\mathfrak{gl}(n,\mathbb{R}) = \{A \mid A \text{ is an } n \times n \text{ matrix}\}.$$

To figure out the Lie bracket operation, we take a matrix $A = (A_{ij})_{n\times n} \in \mathfrak{gl}(n,\mathbb{R})$, and denote the global coordinate system on $\mathrm{GL}(n,\mathbb{R})$ by (X^{ij}). Then the corresponding tangent vector at $T_{I_n}\mathrm{GL}(n,\mathbb{R})$ is $\sum A_{ij}\frac{\partial}{\partial X^{ij}}$, and the corresponding left-invariant vector on G at the matrix $X = (X^{ij})$ is $\sum X^{ik}A_{kj}\frac{\partial}{\partial X^{ij}}$. It follows that the Lie bracket $[A,B]$ between matrices $A, B \in \mathfrak{g}$ is the matrix corresponding to

$$\begin{aligned}\left[\sum X^{ik}A_{kj}\frac{\partial}{\partial X^{ij}}, \sum X^{pq}B_{qr}\frac{\partial}{\partial X^{pr}}\right] &= \sum X^{ik}A_{kj}B_{jr}\frac{\partial}{\partial X^{ir}} - \sum X^{pq}B_{qr}A_{rj}\frac{\partial}{\partial X^{pj}} \\ &= \sum X^{ik}\left(A_{kr}B_{rj} - B_{kr}A_{rj}\right)\frac{\partial}{\partial X^{ij}}.\end{aligned}$$

In other words, the Lie bracket operation on $\mathfrak{g}$ is the matrix commutator

$$[A,B] = AB - BA.$$