LECTURE 18: LIE SUBGROUPS

1. Lie subgroups v.s. Lie subalgebras

We have defined Lie groups and Lie algebras, and studied the correspondence between them (in a primitive level). (We have showed that for any Lie group G, one can associate to it a God-given Lie algebra $\mathfrak{g}$. One deep theorem in Lie theory, which we will not study in this course, claims that for any finite dimensional Lie algebra $\mathfrak{g}$, there is a unique simply connected Lie group G whose Lie algebra is $\mathfrak{g}$. So there is a nice correspondence between Lie groups and Lie algebras.) Now we would like to define sub-objects for Lie groups and Lie algebras, and in particular we would like a correspondence between Lie subgroups and Lie subalgebras.

Of course a Lie subgroup should be both a submanifold and a subgroup, while a Lie subalgebra should be both a vector subspace and a Lie algebra. There is one ambiguity in defining Lie subgroups: should we require a Lie subgroup to be a smooth submanifold, or just an immersed submanifold?

Since we would like a nice correspondence between Lie subgroups of a given Lie group G and Lie subalgebras of the corresponding Lie algebra $\mathfrak{g}$, let's first study Lie subalgebras (whose definition is clear), and then look at the corresponding Lie subgroups.

Definition 1.1. A *Lie subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a vector subspace so that $[X,Y]_{\mathfrak{g}} \in \mathfrak{h}$ holds for all $X, Y \in \mathfrak{h}$.

The simplest example is $\mathfrak{g} = \mathbb{R}^2$, endowed with trivial Lie algebra $[\cdot,\cdot] \equiv 0$. Then any vector subspace of $\mathbb{R}^n$ is a Lie subalgebra of $\mathfrak{g}$. To get a non-trivial phenomena out of this trivial example, we take the corresponding Lie group G to be $G = \mathbb{T}^2$ (instead of the additive group $\mathbb{R}^2$). We take an arbitrary Lie subalgebra of $\mathfrak{g}$, which is a one-dimensional vector subspace in $\mathfrak{g}$, (we omit the 0-dimensional case)

$$\mathfrak{h}_\alpha = \text{the line passing the origin in } \mathbb{R}^2 \text{ whose slope equals } \alpha.$$

To get the corresponding Lie subgroups of G whose Lie algebra is $\mathfrak{h}_a$, we have to distinguish lines with rational slopes with lines with irrational slopes: If $\alpha = p/q$, where p, q are co-prime integers, then the Lie subgroup of G whose Lie algebra equals $\mathfrak{h}_\alpha$ is

$$H^\alpha = H^{p,q} := \{(e^{ipt}, e^{iqt}) \mid t \in \mathbb{R}\}.$$

These subgroups are all isomorphic to S^1 and are all embedded submanifolds in $\mathbb{T}^2$. However, if α is irrational, then the corresponding Lie subgroups are "dense curves" of the form

$$H^\alpha := \{(e^{it}, e^{i\alpha t}) \mid t \in \mathbb{R}\},$$

which are immersed submanifolds in $\mathbb{T}^2$ and are isomorphic to $\mathbb{R}$. Since $\overline{H}^\alpha = \mathbb{T}^2$, they are not embedded submanifolds.

In conclusion, Lie subgroups "should" be defined as immersed submanifolds:

Definition 1.2. A subgroup H of a Lie group G is called a *Lie subgroup* if it is an immersed submanifold, and the group multiplication $\mu_H = \mu_G|_{H\times H}$ is smooth.

So if H is a Lie subgroup of G, then the inclusion $\iota_H : H \hookrightarrow G$ is a Lie group injective homomorphism. Note that we don't require the topology and the smooth structure of a Lie subgroup H of G to be the ones inherited from G.

Remark. According the previous example, a Lie subgroup of a compact Lie group could be non-compact.

Now suppose $\iota_H : H \hookrightarrow G$ a Lie subgroup, and let $\mathfrak{h}$ be the Lie algebra of H. Since $d\iota_H : \mathfrak{h} \to \mathfrak{g}$ is injective, and since

$$d\iota_H([X,Y]^{\mathfrak{h}}) = [d\iota_H(X), d\iota_H(Y)]^{\mathfrak{g}}, \qquad \forall X, Y \in \mathfrak{h},$$

we can identify $\mathfrak{h}$ with $d\iota_H(\mathfrak{h})$ and thus think of $\mathfrak{h}$ as a *Lie subalgebra* of $\mathfrak{g}$. According to the naturality of the exponential map,

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{d\iota_H} & \mathfrak{g} \\ \Big\downarrow \exp_H & & \Big\downarrow \exp_G \\ H & \xrightarrow{\iota_H} & G \end{array}$$

we conclude that $\exp_H : \mathfrak{h} \to H$ is exactly the restriction of $\exp_G : \mathfrak{g} \to G$ onto $\mathfrak{h}$.

Theorem 1.3. *Suppose H is a Lie subgroup of G. Then as a Lie subalgebra of $\mathfrak{g}$,*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \text{ for all } t \in \mathbb{R}\}.$$

Proof. If $X \in \mathfrak{h}$, then for any $t \in \mathbb{R}$,

$$\exp_G(tX) = \exp_H(tX) \in H.$$

Conversely we fix $X \notin \mathfrak{h}$ and Consider the map

$$\varphi : \mathbb{R} \times \mathfrak{h} \to G, \quad (t, Y) \mapsto \exp_G(tX)\exp_G(Y).$$

Using the facts $d\exp_0 = \mathrm{Id}$ and $d\mu_{e,e}(X,Y) = X + Y$, we get

$$d\varphi_{0,0}(\tau, \widetilde{Y}) = \tau X + \widetilde{Y}.$$

Since $X \notin \mathfrak{h}$, $d\varphi_{0,0}$ is injective. It follows that there exists a small $\varepsilon > 0$ and a neighborhood U of 0 in $\mathfrak{h}$ such that φ maps $(-\varepsilon, \varepsilon) \times U$ injectively into G. Shrinking U if necessary, we may assume that $\exp_H$ maps U diffeomorphically onto a neighborhood $\mathcal{U}$ of e in H. Choose a smaller neighborhood $\mathcal{U}_0$ of e in H such that $\mathcal{U}_0^{-1}\mathcal{U}_0 \subset \mathcal{U}$. (The existence of $\mathcal{U}_0$ is guaranteed by PSet 1 Part 1 Problem 1(a)!) We pick a countable collection $\{h_j \mid j \in \mathbb{N}\} \subset H$ such that $h_j\mathcal{U}_0$ cover H. (This is always possible since any open covering of H admits a countable sub-covering.)

For each j denote $T_j = \{t \in \mathbb{R} \mid \exp_G(tX) \in h_j\mathcal{U}_0\}$. We claim that T_j is a countable set. In fact, if $|t - s| < \varepsilon$ and $t, s \in T_j$, then

$$\exp_G(t-s)X = \exp_G(-sX)\exp_G(tX) \in \mathcal{U}.$$

So $\exp_G(t-s)X = \exp_G(Y)$ for a unique $Y \in U$. It follows $\varphi(t-s,0) = \varphi(0,Y)$. Since φ is injective on $(-\varepsilon,\varepsilon)\times U$, we conclude $Y=0$ and $t=s$.

Now each T_j is a countable set. So one can find $t \in \mathbb{R}$ such that $t \notin T_j$ for all j. It follows that $\exp_G(tX) \notin \cup_j h_j \mathcal{U}_0 = H$. So

$$X \notin \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \text{ for all } t \in \mathbb{R}\}.$$

This completes the proof. □

Example. **(The special linear group)** The special linear group is defined as

$$\mathrm{SL}(n,\mathbb{R}) = \{X \in \mathrm{GL}(n,\mathbb{R}) : \det X = 1\}.$$

It is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. To determine its Lie algebra $\mathfrak{sl}(n,\mathbb{R})$, we notice that

$$\det e^A = e^{\mathrm{Tr}(A)}.$$

So for an $n\times n$ matrix A, $e^A \in \mathrm{SL}(n,\mathbb{R})$ if and only if $\mathrm{Tr}(A)=0$. We conclude

$$\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid \mathrm{Tr}(A) = 0\}.$$

Example. **(The orthogonal group)** Next let's consider the *orthogonal group*

$$\mathrm{O}(n) = \{X \in \mathrm{GL}(n,\mathbb{R}) : X^TX = I_n\}.$$

This is a compact $\frac{n(n+1)}{2}$ dimensional Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. To figure out its Lie algebra $\mathfrak{o}(n)$, we note that $(e^A)^T = e^{A^T}$, so

$$(e^{tA})^T e^{tA} = I_n \Longleftrightarrow e^{tA^T} = e^{-tA}.$$

Since exp is locally bijective, we conclude that $A \in \mathfrak{o}(n)$ if and only if $A^T = -A$. So

$$\mathfrak{o}(n) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^T + A = 0\},$$

which is the space of $n\times n$ skew-symmetric matrices.

Notice that $\mathrm{O}(n)$ is not connected. It consists of two connected components, and the connected component of identity is the called the *special orthogonal groups*

$$\mathrm{SO}(n) = \{X \in \mathrm{GL}(n,\mathbb{R}) : X^TX = I_n, \det X = 1\} = \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R}).$$

Its Lie algebra $\mathfrak{so}(n)$ is the same as $\mathfrak{o}(n)$.

Remark. Any Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ is called a *linear Lie group*. According to a deep theorem in Lie theory, any compact Lie group is isomorphic to a linear Lie group. On the other hand, there exist non-compact Lie groups which are not linear Lie groups.

So from any Lie subgroup H of a Lie group G we get a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Next we will show that there is a one-to-one correspondence between Lie subalgebras of $\mathfrak{g}$ and connected Lie subgroups of G:

Theorem 1.4. *Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$.*

Proof. (Idea: Our data is $\mathfrak{h}$, a set of vector fields. We want to construct a Lie subgroup, which is a smooth manifold. How? Use the Frobenius theorem!)

Construct H: Let $X_1, \cdots, X_k$ be a basis of $\mathfrak{h} \subset \mathfrak{g}$. Since $X_i's$ are left invariant vector fields on G, linearly independent at e, they are linearly independent at all $g \in G$. Let

$$\mathcal{V}_g = \text{span}\{X_1(g), \cdots, X_k(g)\}.$$

Then

- $\mathcal{V}$ is a distribution of dimension k on G.
- The distribution $\mathcal{V}$ is a left invariant: $dL_{g_1}(\mathcal{V}_{g_2}) = \mathcal{V}_{g_1 g_2}$. It follows that if N is an integral manifold of $\mathcal{V}$, so is $L_g(N)$ for any g.
- The distribution is involutive, since $[X_i, X_j] \in \mathfrak{h}$ for all i, j.

According to the Frobenius theorem, there is a unique maximal connected integral manifold of $\mathcal{V}$ through e. Denote this by H. It is an immersed submanifold of G.

(Idea: How to prove that an element is contained in the maximal connected integral submanifold H? use: if H' is another integral manifold and $H' \cap H \neq \emptyset$, then $H' \subset H$.)

H is a subgroup: Take any $h_1, h_2 \in H$.

- Since $h_1 = L_{h_1} e \in H \cap L_{h_1} H \neq \emptyset$, and since H is maximal, we have $L_{h_1} H \subset H$. So in particular $h_1 h_2 = L_{h_1} h_2 \in H$.
- Since $L_{h_1^{-1}}(h_1) = e \in H \cap L_{h_1^{-1}} H$, and since H is maximal, we have $L_{h_1^{-1}} H \subset H$. So in particular $h_1^{-1} = L_{h_1^{-1}} e \in H$.

It follows that H is a subgroup of G.

(Idea: The smoothness is a issue since the topology/smooth structure on $H \neq$ that of G. To overcome, use: locally any immersed submanifold is embedded.)

The smoothness of μ_H. First notice that the composition map

$$H \times H \hookrightarrow G \times G \overset{\mu_G}{\to} G$$

is smooth. Let $h_1, h_2 \in H$. Since H is an immersed submanifold of G, we can take small open neighborhoods H_1, H_2, H_3 of h_1, h_2 and $h_1 h_2$ in H respectively, so that each $H_i, i = 1, 2, 3$, is an embedded submanifold of G. Shrinking H_1 and H_2 if necessary, we can assume $H_1 \cdot H_2 \subset H_3$. (Can you write down the details here?) Since the smooth structures on H_i's are the ones inherited from G, we conclude that $\mu_H : H_1 \times H_2 \to H_3$ is smooth. Since h_1, h_2 are arbitrary, μ_H is smooth.

(Idea: How to prove two connected Lie groups are the same? Use: each Lie group is generated by any neighborhood of e.)

Uniqueness of H: Let K be another connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then K is also an integral manifold of $\mathcal{V}$ passing e. So we have $K \subset H$. Since $T_e K = T_e H$, the inclusion has to be a local diffeomorphism near e. In other words, there exist an open subset K_e of K near e and an open subset H_e of H near e so that $K_e = H_e$. By connectedness of K and H, we get $K = \cup_{j \geq 1} K_e^j = \cup_{j \geq 1} H_e^j = H$. □

2. Closed Lie Subgroups (Reading Material)

We are most interested in those Lie subgroups H of G that are not just immersed submanifolds, but in fact smooth submanifolds.

Proposition 2.1. *Suppose G is a Lie group, H is a subgroup of G which is a submanifold as well. Then H is a closed subset in G.*

Proof. Since H is a submanifold of G, it is *locally closed* everywhere. In particular, one can find an open neighborhood U of e in G such that $U \cap H = U \cap \overline{H}$. Now take any $h \in \overline{H}$. Since hU is an open neighborhood of h in G, $hU \cap H \neq \emptyset$. Let $h' \in hU \cap H$, then $h^{-1}h' \in U$. On the other hand, since $h \in \overline{H}$, there is a sequence h_n in H converging to h. It follows that the sequence $h_n^{-1}h' \in H$ converges to $h^{-1}h'$. In other words, $h^{-1}h' \in U \cap \overline{H} = U \cap H$. So $h \in H$. This completes the proof. □

Definition 2.2. A subgroup H of a Lie group G is called a *closed subgroup* if it is a closed subset in G. (Here we don't require H to be a Lie subgroup!)

So if H is a Lie subgroup of G and is a smooth submanifold, then H is a closed subgroup. In what follows we will prove the following very powerful theorem, which claims that the converse is true:

Theorem 2.3 (E. Cartan's closed subgroup theorem)**.** *Any closed subgroup H of a Lie group G is a Lie subgroup (and thus a smooth submanifold) of G.*

Remark. In this theorem we only assume that *algebraically* H is a subgroup, *topologically* H is a closed subset, and we arrive at the strong conclusion that H is a Lie subgroup (and in particular H itself is a Lie group). This is a very powerful tool to prove various groups are Lie groups. For example, $O(n)$, $SL(n,\mathbb{R})$ are Lie groups, since they are closed subgroups of $\mathrm{GL}(n,\mathbb{R})$.

As an immediate consequence of Cartan's theorem, we get

Corollary 2.4. *If $\varphi : G \to H$ is Lie group homomorphism, then $\ker(\varphi)$ is a Lie subgroup of G.*

Now we proceed to prove Theorem 2.3. Suppose H is a closed subgroup of G. Let

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp(tX) \in H \text{ for all } t \in \mathbb{R}\}.$$

We will need the following lemmas:

Lemma 2.5. *$\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$.*

Proof. Clearly $\mathfrak{h}$ is closed under scalar multiplication. We have seen that there exists smooth function $Z : (-\varepsilon, \varepsilon) \to \mathfrak{g}$ so that

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + t^2 Z(t)).$$

It follows

$$\exp(t(X+Y)) = \lim_{n\to\infty} \left(\exp(\frac{tX}{n}) \exp(\frac{tY}{n}) \right)^n.$$

So by closedness of H, $\mathfrak{h}$ is also closed under addition. □

Lemma 2.6. *Endow with* $\mathfrak{g}$ *an inner product. Suppose* $X_1, X_2, \cdots \in \mathfrak{g}$ *satisfy*

(1) $X_i \neq 0$ *and* $X_i \to 0$ *as* $i \to \infty$.
(2) $\exp(X_i) \in H$ *for all* i.
(3) $\lim_{i\to\infty} \frac{X_i}{|X_i|} = X \in \mathfrak{g}$.

Then $X \in \mathfrak{h}$.

Proof. Fix any $t \neq 0$. Let $n_i = \left[\frac{t}{|X_i|}\right]$ be the integer part of $\frac{t}{|X_i|}$. Then

$$|n_i X_i - tX| \leq \left|\left[\frac{t}{|X_i|}\right] - \frac{t}{|X_i|}\right| |X_i| + t\left|\frac{X_i}{|X_i|} - X\right| \to 0.$$

So

$$\exp(tX) = \lim_{i\to\infty} \exp(n_i X_i) = \lim_{i\to\infty} (\exp X_i)^{n_i} \in H.$$

□

Lemma 2.7. *The exponential map* $\exp : \mathfrak{g} \to G$ *maps a neighborhood of* 0 *in* $\mathfrak{h}$ *bijectively to a neighborhood of* e *in* H.

Proof. Take a vector subspace $\mathfrak{h}'$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$. Define $\Phi : \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}' \to G$ by

$$\Phi(X + Y) = \exp(X)\exp(Y).$$

Then $d\Phi_0(\widetilde{X} + \widetilde{Y}) = \widetilde{X} + \widetilde{Y}$. So near 0, Φ is a local diffeomorphism from $\mathfrak{g}$ to G. Since $\exp|_{\mathfrak{h}} = \Phi|_{\mathfrak{h}}$, to prove the lemma, it is enough to prove that Φ maps a neighborhood of 0 in $\mathfrak{h}$ bijectively to a neighborhood of e in H.

Suppose the lemma is false, then we can find $X_i + Y_i \in \mathfrak{h} \oplus \mathfrak{h}'$ with $Y_i \neq 0$ so that $X_i + Y_i \to 0$ and $\Phi(X_i + Y_i) \in H$. Since $\exp(X_i) \in H$, we must have $\exp(Y_i) \in H$ for all i. We let Y be a limit point of $\frac{Y_i}{|Y_i|}{}'s$. Then by Lemma 2.6, $Y \in \mathfrak{h}$. Since $Y \in \mathfrak{h}'$, we must have $Y = 0$, which is a contradiction since by construction, $|Y| = 1$. □

Proof of Cartan's closed subgroup theorem. According to Lemma 2.7, one can find a neighborhood U of e in G and a neighborhood V of 0 in $\mathfrak{g}$ so that $\exp^{-1} : U \to V$ is a diffeomorphism, and so that $\exp^{-1}(U \cap H) = V \cap \mathfrak{h}$. It follows that $(\exp^{-1}, U, V)$ is a chart on G which makes H a submanifold near e. For any other point $h \in H$, we can use left translation to get such a chart. This proves that H is a smooth submanifold of G. The smoothness of μ_H is obvious. □

Another very interesting consequence of Cartan's theorem is

Corollary 2.8. *Every continuous homomorphism between Lie groups is smooth.*

Proof. Let $\phi : G \to H$ be a continuous homomorphism, then $\Gamma_\phi = \{(g, \phi(g)) \mid g \in G\}$ is a closed subgroup, and thus a Lie subgroup of $G \times H$. It follows that the projection

$$p : \Gamma_\phi \overset{i}{\hookrightarrow} G \times H \overset{\pi_1}{\longrightarrow} G$$

is bijective and smooth. Moreover, $dp_{(e_G, e_H)}$ is bijective. So p is local diffeomorphism near (e_G, e_H). By left translation, p is local diffeomorphism everywhere. Since p is bijective, it has to be a global diffeomorphism. Thus $\phi = \pi_2 \circ p^{-1}$ is smooth. □