LECTURE 21: DIFFERENTIAL FORMS

1. Differential Forms

Let M be a smooth manifold. We have associated to each $p \in M$ a vector space T_pM. If we take any local chart (φ, U, V) around p, then we can write down an explicit basis for T_pM:

$$\partial_i|_p : C^\infty(U) \to \mathbb{R}, \quad \partial_i|_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)), \qquad (1 \le i \le n).$$

Note that not only the $\partial_i|_p$'s form a basis for the tangent space T_pM, but in fact ∂_i's are smooth vector fields on U, and for any $q \in U$, the $\partial_i|_q$'s form a basis for the tangent space T_qM.

Now let's study the dual space T_p^*M of T_pM. It is called the *cotangent space* of M at p, and elements in T_p^*M are called *cotangent vectors* at p. It is also quit easy to write down an explicit basis of T_q^*M, (and in fact a basis of T_q^*M, for each $q \in U$, varying smoothly in q), in any given local chart (φ, U, V): We first note that for each $1 \le i \le n$,

$$x^i \circ \varphi : U \to \mathbb{R}$$

is a smooth function on U. The differential of this function, which we will denote by dx^i for simplicity, is a linear map (when restricted to any $q \in U$)

$$dx^i|_q : T_qM = T_qU \to T_{x^i \circ \varphi(q)}\mathbb{R} = \mathbb{R}.$$

In other words, each $dx^i|_q$ is an element in T_q^*M. More over, by definition,

$$dx^i|_q(\partial_j|_q) = \partial_j|_q(x^i \circ \varphi) = \delta_j^i.$$

So we conclude

Proposition 1.1. *In any local chart (φ, U, V), $\{dx^i|_q : 1 \le i \le n\}$ is a basis of T_q^*M. Moreover, this basis is the dual basis to the basis $\{\partial_i|_q : 1 \le i \le n\}$ of T_qM.*

In fact, for any $f \in C^\infty(U)$, by the same way we get a linear map $df_q : T_qM \to \mathbb{R}$. In other words, we get a cotangent vector $df_q \in T_q^*M$. By definition, $df_p(\partial_i|_p) = \partial_i|_p(f)$. It follows

$$df_p = (\partial_1|_p f)dx^1|_p + \cdots + (\partial_n|_p f)dx^n|_p,$$

and moreover, for any $X \in \Gamma^\infty(TU)$,

$$df(X) = Xf,$$

where both sides are regarded as functions on U.

Now we are ready to define (Compare: the definition of vector fields on manifolds)

Definition 1.2. An (l,k)-*tensor field* T on M is an assignment that assigns to each point $p \in M$ an (l,k)-tensor $T_p \in \otimes^{l,k}T_pM$.

Remark. By definition, T is a tensor at if and only if it is point-wise linear in each entry. It follows that T is tensor field on M if and only if it is "function-linear" in each entry. So it is more than a multi-linear map, i.e., we also have (where ω's are 1-forms on U, and X's are vector fields on U)

$$T(f_1\omega^1, \cdots, f_l\omega^l, g^1X_1, , \cdots, g^kX_k) = f_1 \cdots f_l g^1 \cdots g^k T(\omega_1, \cdots, \omega_l, X_1, \cdots, X_k).$$

If we fix any local chart (φ, U, V) near p, then we can write

$$T_p = \sum T^{i_1 \cdots i_l}_{j_1 \cdots j_k}\ \partial_{i_1}|_p \otimes \cdots \otimes \partial_{i_l}|_p \otimes dx^{j_1}|_p \otimes \cdots \otimes dx^{j_k}|_p,$$

where $T^{i_1 \cdots i_l}_{j_1 \cdots j_k}$'s are constants (which depends on p). In other words, in any coordinate chart U one can write

$$T = \sum T^{i_1 \cdots i_l}_{j_1 \cdots j_k} \partial_{i_1} \otimes \cdots \otimes \partial_{i_l} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_k},$$

where $T^{i_1 \cdots i_l}_{j_1 \cdots j_k}$'s are functions on U.

Definition 1.3. We say a (l, k)-tensor field T on M is *smooth* if in any coordinate chart U, the functions $T^{i_1 \cdots i_l}_{j_1 \cdots j_k}$'s are smooth.

Note that when $(l, k) = (0, 1)$, we will get a smooth vector field on M. The set of all smooth (l, k)-tensors is denoted by $\Gamma^\infty(\otimes^{l,k}TM)$. Again this is an infinite dimensional vector space.

Remark. The coefficient functions $T^{i_1 \cdots i_l}_{j_1 \cdots j_k}$'s are only defined in local charts. If one uses another chart U', one gets another set of coefficient functions (even if at the same point).

Example. A symmetric positive smooth $(0, 2)$-tensor field g on M is called a *Riemannian metric* on M. Locally a Riemannian metric is of the form

$$g = \sum g_{ij}(x) dx^i \otimes dx^j,$$

where $(g_{ij}(x))$ is a positive definite symmetric matrix depending smoothly on x.

Similarly one can define smooth k-forms on a smooth manifold M:

Definition 1.4. A *k-form* ω on a smooth manifold M is an assignment that assigns to each point $p \in M$ a linear k-form $\omega_p \in \Lambda^k T_p^* M$. A k-form ω is *smooth* if locally one can write

$$\omega = \sum_I \omega_I dx^I = \sum_I \omega_{i_1, \cdots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the summation is over increasing k-tuples $I = \{1 \le i_1 < \cdots < i_k \le n\}$, and each $\omega_I \in C^\infty(U)$.

Since k-forms will be frequently used in the rest of this course, we will denote the set of all smooth k-forms by $\Omega^k(M)$ (instead of the lengthy expression $\Gamma^\infty(\Lambda^k T^* M)$). Note that any smooth function on M can be viewed as a smooth 0-form. So

$$\Omega^0(M) = C^\infty(M).$$

Since there is no linear k-form on T_pM for $k > n = \dim M$, we get

$$\Omega^k(M) = 0, \quad \forall k > n.$$

Note that if $\omega \in \Omega^k(M)$, and $X_1, \cdots, X_k \in \Gamma^\infty(TM)$, then $\omega(X_1, \cdots, X_k) \in C^\infty(M)$.

2. The exterior derivative

Now we define the exterior derivative for differential forms. It generalizes the conception of the differential on functions, and it is the most important operation for the rest of this semester. Unlike the wedge product, the interior product and the pull-back operations that we defined last time, the exterior derivative is no longer a pointwise operation, but is a local operation (i.e. depends on the "nearby values")

We start with $f \in \Omega^0(U) = C^\infty(U)$. In this case we have already seen that $df \in \Omega^1(U)$. So we get a linear map

$$d : \Omega^0(U) \to \Omega^1(U), \qquad f \mapsto df.$$

Locally on each coordinate chart we have

$$df = \sum_i (\partial_i f) dx^i.$$

We also have an "invariant definition" of $df \in \Omega^1(U)$, via

$$df(X) = Xf, \quad \forall X \in \Gamma^\infty(TU).$$

Now suppose ω is a k-form on M, so that locally

$$\omega = \sum_I \omega_{i_1,\cdots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We want to define $d\omega$ as a $(k+1)$-form. It is natural to define

Definition 2.1. The *exterior derivative* of ω is the $(k+1)$-form $d\omega$ given by the formula

$$\begin{aligned} d\omega &= \sum_I d\omega_{i_1,\cdots,i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{I,i} \partial_i(\omega_{i_1,\cdots,i_k}) dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{aligned} \tag{1}$$

Before we proceed, we need to clarify that $d\omega$ defined above is well-defined. In other words, the $(k+1)$-form $d\omega$ defined above should be independent of the choices of coordinate patches.

Usually one has two ways to prove the well-definedness of a definition on manifolds. One is to check that the definition is unchanged if one use another coordinate chart (for example, the definition of smoothness of a function), the other is to give an equivalent but coordinate-free definition (usually called *the invariant formulation*). We will take the second approach here since the coordinate-free expression of $d\omega$ is also very useful.

Let's start with small k's to find out the invariant formula of $d\omega$.

- For $k = 0$, i.e. $\omega = f \in C^\infty(U)$, we can regard df as a $C^\infty(U)$-linear map

  $$df : \Gamma^\infty(TU) \to C^\infty(U)$$

  such that

  $$df(X) = Xf.$$

- For $k = 1$, i.e. $\omega \in \Omega^1(U)$, we want to regard $d\omega$ as a $C^\infty(U)$-bilinear map
$$d\omega : \Gamma^\infty(TU) \times \Gamma^\infty(TU) \to C^\infty(U).$$
We write $\omega = \sum_i \omega_i dx^i$, $X = \sum_k X^k \partial_k$ and $Y = \sum_l Y^l \partial_l$. Then
$$\begin{aligned} d\omega(X,Y) &= \sum_{i,j,k,l} (\partial_j \omega_i) dx^j \wedge dx^i (X^k \partial_k, Y^l \partial_l) \\ &= \sum_{i,j} \big((\partial_j \omega_i) X^j Y^i - (\partial_j \omega_i) X^i Y^j \big) \\ &= \sum_{i,j} \big(X^j \partial_j (\omega_i Y^i) - \omega_i X^j \partial_j (Y^i) - Y^j \partial_j (\omega_i X^i) + \omega_i Y^j \partial_j (X^i) \big) \\ &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]). \end{aligned}$$
So we arrive at
$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$
- For $k = 2$, i.e. $\omega \in \Omega^2(U)$, by a tedious computation one can prove that as a $C^\infty(U)$-trilinear map
$$d\omega : \Gamma^\infty(TU) \times \Gamma^\infty(TU) \times \Gamma^\infty(TU) \to C^\infty(U),$$
one has
$$\begin{aligned} d\omega(X,Y,Z) =& X(\omega(Y,Z)) - Y(\omega(X,Z)) + Z(\omega(X,Y)) \\ & - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z].X). \end{aligned}$$

So we are naturally led to the following *the invariant formula* for $d\omega$:

Theorem 2.2. *For any $\omega \in \Omega^k(U)$, the $(k+1)$-form $d\omega$, viewed as a $C^\infty(U)$-multilinear map*
$$d\omega : \Gamma^\infty(TU) \times \cdots \times \Gamma^\infty(TU) \to C^\infty(U),$$
is given by the formula
$$\begin{aligned} d\omega(X_1, \cdots, X_{k+1}) :=& \sum_i (-1)^{i-1} X_i (\omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{k+1})) \\ & + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{k+1}). \end{aligned} \tag{2}$$

Sketch of proof. Define $d\omega$ via the formula (2). We need to show that

(1) $d\omega$ is anti-symmetric, i.e. for any $r < s$,
$$d\omega(X_1, \cdots, X_r, \cdots, X_s, \cdots, X_{k+1}) = -d\omega(X_1, \cdots, X_s, \cdots, X_r, \cdots, X_{k+1}).$$
This follows from a simple but messy computation. (But you can get the idea by staring at the formulae for $d\omega$ with $\omega \in \Omega^1(U)$ or $\omega \in \Omega^2(U)$ above.)

(2) $d\omega$ is multi-linear at each point, i.e. $d\omega$ is $C^\infty(U)$-linear on U. Note that $d\omega$ is obviously $\mathbb{R}$-linear. So in view of (1), it is enough to prove for any $f \in C^\infty(U)$,

$$d\omega(fX_1, X_2, \cdots, X_{k+1}) = f d\omega(X_1, \cdots, X_{k+1}).$$

This can be checked by a direct computation:

$$\begin{aligned} d\omega(fX_1, X_2, \cdots, X_{k+1}) =& fX_1(\omega(X_2, \cdots, X_{k+1})) \\ &+ \sum_{i>1} (-1)^{i-1} X_i(\omega(fX_1, \cdots, \widehat{X_i}, \cdots, X_{k+1})) \\ &+ \sum_{i>1} (-1)^{i+1} \omega([fX_1, X_i], X_2, \cdots, \widehat{X_i}, \cdots, X_{k+1}) \\ &+ \sum_{1<i<j} (-1)^{i+j} \omega([X_i, X_j], fX_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{k+1}) \\ =& f d\omega(X_1, \cdots, X_{k+1}) \\ &+ \sum_{i>1} (-1)^{i-1} (X_i f) \omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{k+1}) \\ &- \sum_{i>1} (-1)^{i+1} (X_i f) \omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{k+1}) \\ =& f d\omega(X_1, \cdots, X_{k+1}). \end{aligned}$$

(3) It remains to check that $d\omega$ has the local expression (1) as we anticipated. Obviously the map

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

is linear. So without loss of generality, we may assume

$$\omega = f dx^1 \wedge \cdots \wedge dx^k$$

in a local chart U. Note that $[\partial_i, \partial_j] = 0$. It follows that for any increasing indices $j_1 < \cdots < j_{k+1}$, the right hand side of

$$d\omega(\partial_{j_1}, \cdots, \partial_{j_{k+1}}) = \sum_i (-1)^{i-1} \partial_{j_i}(\omega(\partial_{j_1}, \cdots, \hat{\partial}_{j_i}, \cdots, \partial_{j_{k+1}}))$$

vanishes except for the case $j_1 = 1, \cdots, j_k = k$ and $i = k+1$. In other words, the only non-zero terms in all possible expressions $d\omega(\partial_{j_1}, \cdots, \partial_{j_{k+1}})$ are

$$d\omega(\partial_1, \cdots, \partial_k, \partial_r) = (-1)^k \partial_r(f).$$

It follows that

$$d\omega = \sum_{r>k} (-1)^k \partial_r(f) dx^1 \wedge \cdots \wedge dx^k \wedge dx^r = \sum \partial_r(f) dx^r \wedge dx^1 \wedge \cdots \wedge dx^k,$$

which is exactly the local expression we want.

□

3. Relations between various operations on differential forms

Of course the pointwise operations for linear k-forms that we learned last time still make sense for differential forms on manifolds. So on differential forms we have the following operations:

- The *wedge product* $\wedge : \Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$.
  - For example,
  $$(dx^1 + 2dx^2) \wedge (dx^1 \wedge dx^2 - dx^2 \wedge dx^3 + 3dx^1 \wedge dx^3) = -7dx^1 \wedge dx^2 \wedge dx^3.$$
- For any $X \in \Gamma^\infty(TU)$, one has the *interior product* $\iota_X : \Omega^k(U) \to \Omega^{k-1}(U)$.
  - For example,
  $$\iota_X dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_r (-1)^{r-1} dx^{i_r}(X) dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_r}} \wedge \cdots \wedge dx^{i_k}.$$
- For any smooth map $\varphi : U' \to U$, one has the *pull-back* $\varphi^* : \Omega^k(U) \to \Omega^k(U')$.
  - This is defined pointwise via the linear map $d\varphi_p : T_pU' \to T_{\varphi(p)}U$. So if $\omega \in \Omega^k(U)$, then
  $$(\varphi^*\omega)_p(X_1, \cdots, X_k) = \omega_{\varphi(p)}(d\varphi_p(X_1), \cdots, d\varphi_p(X_k)).$$
  Note: if $k = 0$, then φ^* is exactly the pull-back $\varphi^* : C^\infty(U) \to C^\infty(U')$ on functions.
- The *exterior derivative* $d : \Omega^k(U) \to \Omega^{k+1}(U)$.

These operations are all linear (where $\wedge$ is bilinear). Here we list some important properties of these operations.

Proposition 3.1. *Suppose $\omega \in \Omega^k(U), \eta \in \Omega^l(U)$, $X \in \Gamma^\infty(TU)$ and $\varphi \in C^\infty(U', U)$. Then*

(1) $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.
(2) $\varphi^*(\omega \wedge \eta) = \varphi^*\omega \wedge \varphi^*\eta$.
(3) $\iota_X(\omega \wedge \eta) = (\iota_X\omega) \wedge \eta + (-1)^k \omega \wedge \iota_X\eta$.
(4) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

Suppose $X \in \Gamma^\infty(TU)$, $\varphi \in C^\infty(U', U)$ and $\psi \in C^\infty(U, \widetilde{U})$. Then

(5) $\iota_X \circ \iota_X = 0$.
(6) $d \circ d = 0$.
(7) $(\psi \circ \varphi)^* = \psi^* \circ \varphi^*$.
(8) $\varphi^* \circ d = d \circ \varphi^*$.

Proof. (1), (2), (3), (5), (7) follows from definitions and from corresponding results in last lecture.

(4): Since d is linear, it is enough to assume $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\eta = g dx^{j_1} \wedge \cdots \wedge dx^{j_l}$, with indices set $I \cap J = \emptyset$. Then the formula follows from a direct computation.

(6): We first check this for $k = 0$:
$$d(df)(X, Y) = X(df(Y)) - Y(df(X)) - df([X, Y]) = X(Y(f)) - Y(X(f)) - [X, Y]f = 0.$$
For $k > 0$, by linearity we may assume $\omega = f dx^1 \wedge \cdots \wedge dx^k$. Since $ddf = 0$ and $ddx^i = 0$, we get
$$d(d\omega) = d(df \wedge dx^1 \cdots \wedge dx^k) = d(df) \wedge dx^1 \wedge \cdots \wedge dx^k + \sum_i (-1)^i df \wedge dx^1 \wedge \cdots \wedge d(dx^i) \wedge \cdots \wedge dx^k = 0.$$

(8): Again we first check this for $k = 0$:

$$(\varphi^* df)_p(X_p) = df_{\varphi(p)}(d\varphi_p(X_p)) = d(\varphi^* f)_p(X_p).$$

In general, we may assume $\omega = f dx^1 \wedge \cdots \wedge dx^k$. Then by (2) and (6),

$$\begin{aligned}
\varphi^* d\omega &= \varphi^*(df \wedge dx^1 \wedge \cdots \wedge dx^k) \\
&= \varphi^*(df) \wedge \varphi^*(dx^1) \wedge \cdots \wedge \varphi^*(dx^k) \\
&= d(\varphi^* f) \wedge d(\varphi^* x^1) \wedge \cdots \wedge d(\varphi^* x^k) \\
&= d(\varphi^* f d(\varphi^* x^1) \wedge \cdots \wedge d(\varphi^* x^k)) \\
&= d(\varphi^* \omega).
\end{aligned}$$

□