LECTURE 23: THE STOKES FORMULA

1. Volume Forms

Last time we introduced the conception of orientability via local charts. Here is a criteria for the orientability via top forms:

Theorem 1.1. *An n-dimensional smooth manifold M is orientable if and only if M admits a nowhere vanishing smooth n-form μ.*

Proof. First let μ be a nowhere vanishing smooth n-form on M. Then on each local chart $(U, x^1, \cdots, x^n)$ (where U is always chosen to be connected), there is a smooth function $f \neq 0$ so that $\mu = f dx^1 \wedge \cdots \wedge dx^n$. It follows that

$$\mu(\partial_1, \cdots, \partial_n) = f \neq 0.$$

We can always take such a chart near each point so that $f > 0$, otherwise we can replace x^1 by $-x^1$. Now suppose $(U_\alpha, x_\alpha^1, \cdots, x_\alpha^n)$ and $(U_\beta, x_\beta^1, \cdots, x_\beta^n)$ be two such charts, so that on the intersection $U_\alpha \cap U_\beta$ one has

$$\mu = f dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = g dx_\beta^1 \wedge \cdots \wedge dx_\beta^n,$$

where $f, g > 0$. Then on the intersection $U_\alpha \cap U_\beta$,

$$0 < g = \mu(\partial_1^\beta, \cdots, \partial_n^\beta) = (\det d\varphi_{\alpha\beta})\mu(\partial_1^\alpha, \cdots, \partial_\alpha^n) = (\det d\varphi_{\alpha\beta}) f.$$

It follows that $\det(d\varphi_{\alpha\beta}) > 0$. So the atlas constructed by this way is an orientation.

Conversely, suppose $\mathcal{A}$ is an orientation. For each local chart U_α in $\mathcal{A}$, we let

$$\mu_\alpha = dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n.$$

Pick a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. We claim that

$$\mu := \sum_\alpha \rho_\alpha \mu_\alpha$$

is a nowhere vanishing smooth n-form on M. In fact, for each $p \in M$, there is a neighborhood U of p so that the sum $\sum_\alpha \rho_\alpha \mu_\alpha$ is a finite sum $\sum_{i=1}^k \rho_i \mu_i$. It follows that near p,

$$\mu(\partial_1^1, \cdots, \partial_n^1) = \sum_i (\det d\varphi_{1k})\rho_i > 0.$$

So $\mu \neq 0$ near p. $\square$

Definition 1.2. A nowhere vanishing smooth n-form μ on an n-dimensional smooth manifold M is called a *volume form.*

Remark. If M is orientable, and μ is a volume form, then the two orientations of M are represented by μ and $-\mu$ respectively. We denote the two orientations by $[\mu]$ and $[-\mu]$.

Remark. Let μ be a volume form on M, and the orientation on M is chosen to be $[\mu]$. Then we can define a linear functional

$$I : C_c(M) \to \mathbb{R}, \quad f \mapsto \int_M f\mu.$$

(Here, $C_C(M)$ represents the space of continuous functions with compact supports on M. Obviously the integrals above still make sense even if f is not smooth.) Since the orientation on M is chosen to be $[\mu]$, we see the functional I is positive, i.e. $I(f) \geq 0$ for $f \geq 0$. Since any manifold is both locally compact and σ-compact, the Riesz representation theorem implies that there exists a unique Radon measure (=a locally finite, regular measure defined on all Borel sets) m_μ such that

$$I(f) = \int_M f dm_\mu.$$

Using the measure $d\mu$, one can define spaces like $L^p(M, \mu)$.

Remark. In particular, on any Lie group, one can define conceptions like left-invariant differential forms. Since any Lie group is orientable (See Problem Set 6), there exists left-invariant volume form on any Lie group G. The measures associated to left-invariant volume form on Lie groups are called *Haar measures*.

2. The Stokes' formula

The Stokes formula is one of the most important formulae in calculus. We now extend it to smooth manifolds. We need some knowledge of manifolds with boundary. Denote

$$\mathbb{R}^n_+ = \{(x^1, \cdots, x^n) \mid x^n \geq 0\}.$$

The following definition is natural:

Definition 2.1. A topological space M is called an n-dimensional *topological manifold with boundary* if it is Hausdorff, second-countable, and for any $p \in M$, there is a neighborhood U of p which is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

Let M be a topological manifold with boundary, then we can define the *boundary* of M to be

$$\partial M = \{p \in M \mid p \text{ has no neighborhood in } M \text{ that is homeomorphic to } \mathbb{R}^n\}.$$

We will call

$$\text{int}(M) = M \setminus \partial M$$

the *interior* of M. As in Lecture 2, smooth structures on manifolds with boundary can be defined to be atlases so that the transition maps between each pairs of charts are smooth. (Note: A map defined on a subset in $\mathbb{R}^n_+ \subset \mathbb{R}^n$ is smooth if it has a smooth extension to an open neighborhood in $\mathbb{R}^n$. See the middle of page 3 in Lecture 4. By this way, we can talk about the smoothness of transition maps between charts of either types.)

Example. The closed ball

$$B^n(1) = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

is a smooth manifold with boundary, and its boundary is $\partial B^n(1) = S^{n-1}$.

Example. More generally, let M be any smooth manifold, and $f \in C^\infty(M)$. If a is a regular value of f, then the sub-level set

$$M_a = f^{-1}((-\infty, a))$$

is a smooth manifold with boundary.

In the above example, the boundary $\partial M_a = f^{-1}(a)$ is a smooth manifold since a is a regular value. In fact this is always the case.

Lemma 2.2. *Suppose M is a smooth manifold with boundary, then ∂M is a smooth manifold.*

Sketch of proof. Let $(U, x^1, \cdots, x^n)$ be a chart near $p \in \partial M$ that is homeomorphic to $\mathbb{R}^n_+$. Then

$$U \cap \partial M = \{(x^1, \cdots, x^n) \mid x^n = 0\}.$$

Then $(U \cap \partial M, x^1, \cdots, x^{n-1})$ is a chart on ∂M. □

As in the case of smooth manifolds (without boundary), one can define an orientation on a smooth manifold with boundary to be an atlas $\mathcal{A}$ so that $\det(d\varphi_{\alpha\beta}) > 0$ for any two charts $U_\alpha, U_\beta \in \mathcal{A}$. We now prove

Theorem 2.3. *If M is an orientable smooth manifold with boundary of dimension n, then the boundary ∂M is an orientable $n-1$ dimensional submanifold of M.*

Proof. Let $(U_\alpha, x^1_\alpha, \cdots, x^n_\alpha)$ and $(U_\beta, x^1_\beta, \cdots, x^n_\beta)$ be two orientation-compatible charts of M near $p \in \partial M$ so that $M \cap U_\alpha$ is characterized by $x^n_\alpha \geq 0$, and $M \cap U_\beta$ is characterized by $x^n_\beta \geq 0$. We would like to show that the coordinate charts $(U_\alpha \cap \partial M, x^1_\alpha, \cdots, x^{n-1}_\alpha)$ and $(U_\beta \cap \partial M, x^1_\beta, \cdots, x^{n-1}_\beta)$ of ∂M are orientation-compatible. In fact, if we denote the transition map $\varphi_{\alpha\beta}$ between U_α and U_β by $(\varphi^1, \cdots, \varphi^n)$, then on $\partial M \cap U_\alpha \cap U_\beta$, we have $x^n_\alpha = x^n_\beta = 0$. In other words,

$$\varphi^n(x^1, \cdots, x^{n-1}, 0) = 0$$

on $U_\alpha \cap U_\beta \cap \partial M$, and

$$\varphi^n(x^1, \cdots, x^{n-1}, x^n) > 0$$

on $U_\alpha \cap U_\beta \cap \mathrm{int}(M)$, i.e. for x^n¿0. It follows

$$\frac{\partial \varphi^n}{\partial x^i}(x^1, \cdots, x^{n-1}, 0) = 0, \quad i = 1, \cdots, n-1$$

and

$$\frac{\partial \varphi^n}{\partial x^n}(x^1, \cdots, x^{n-1}, 0) \geq 0.$$

Since $(U_\alpha, x^1_\alpha, \cdots, x^n_\alpha)$ and $(U_\beta, x^1_\beta, \cdots, x^n_\beta)$ are orientation-compatible, we have $\det(\frac{\partial \varphi^i}{\partial x^j}) > 0$ everywhere in $U_\alpha \cap U_\beta$. In particular,

$$\det\left(\frac{\partial \varphi^i}{\partial x^j}(x^1,\cdots,x^{n-1},0)\right) = \det\begin{pmatrix} \left(\frac{\partial \varphi^i}{\partial x^j}(x^1,\cdots,x^{n-1},0)\right)_{1\leq i,j\leq n-1} & * \\ 0 & \frac{\partial \varphi^n}{\partial x^n}(x^1,\cdots,x^{n-1},0) \end{pmatrix} > 0.$$

It follows that
$$\det\left(\frac{\partial\varphi^i}{\partial x^j}(x^1,\cdots,x^{n-1},0)\right)_{1\le i,j\le n-1}>0.$$
So the charts $(U_\alpha\cap\partial M, x^1_\alpha,\cdots,x^{n-1}_\alpha)$ and $(U_\beta\cap\partial M, x^1_\beta,\cdots,x^{n-1}_\beta)$ of ∂M are orientation-compatible. $\square$

Remark. The boundary of a non-orientable manifold could be orientable (e.g. the Mobiüs band) or non-orientable (e.g. $[0,1]\times M$, where M is non-orientable).

Remark. Not every result for smooth manifold can be extended to smooth manifold with boundary. For example, if M_1 and M_2 are both smooth manifold with boundary, then $M_1\times M_2$ fails to be a smooth manifold with boundary.

Now let M be an orientable smooth manifold with boundary, and let $[\mu]$ be an orientation on M. The *induced orientation* on ∂M is defined as follows: Suppose locally near the boundary, M is given by $x^n\ge 0$, and such that $\mu=f dx^1\wedge\cdots\wedge dx^n$ with $f>0$. Then locally we define the induced orientation on ∂M to be the one that is represented by the differential form
$$\eta=(-1)^n dx^1\wedge\cdots\wedge dx^{n-1}.$$
By choosing a P.O.U., one can glue these local $(n-1)$-forms on ∂M into a global $(n-1)$-form on ∂M. One can check that $\eta\neq 0$ everywhere. (Please check!!) So η is a volume form which defines an orientation on ∂M.

Note that in coordinate charts near boundary, M is defined by $x^n\ge 0$. So $-x^n$ is the "outward pointing direction" of M, and this boundary orientation is chosen so that
$$d(-x^n)\wedge(-1)^n dx^1\wedge\cdots\wedge dx^{n-1}=dx^1\wedge\cdots\wedge dx^n.$$
In other words, if we want to integrate $\omega\in\Omega^{k-1}(\partial M)$ on ∂M, then locally we need to write
$$\omega=f(x^1,\cdots,x^{n-1})\cdot(-1)^n dx^1\wedge\cdots\wedge dx^{n-1}$$
in a chart (φ,U,V) of ∂M, and then calculate
$$\int_U\omega:=\int_V f(x^1,\cdots,x^{n-1})dx^1\cdots dx^{n-1}.$$

Finally we can state and prove the main theorem:

Theorem 2.4 (Stokes' theorem)**.** *Let M be a smooth orientable n-dimensional manifold with boundary ∂M (with the induced orientation above). For any $\omega\in\Omega^{n-1}(M)$ with compact support, we have*
$$\int_{\partial M}\iota^*_{\partial M}\omega=\int_M d\omega,$$
where $\iota_{\partial M}:\partial M\to M$ is the inclusion map.

Remark. The Theorem holds for manifold without boundary, in which case $\partial M=\emptyset$, so that the left hand side is zero.

Proof. We will discuss the following three cases:

(1) ω is supported in a chart U that is diffeomorphic to $\mathbb{R}^n$.
(2) ω is supported in a chart U that is diffeomorphic to $\mathbb{R}^n_+$.
(3) The general case.

Case (1): Since $\omega = 0$ on ∂M, we have $\int_{\partial M} \iota_{\partial M}^* \omega = 0$.

To calculate $\int_M d\omega$, we denote

$$\omega = \sum_i (-1)^{i-1} f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

where f_i's are compactly supported smooth functions. Then

$$d\omega = \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n.$$

So by definition,

$$\int_M d\omega = \int_{\mathbb{R}^n} \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^n = \sum_i \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n = 0.$$

Case (2): We have the same formula to calculate $\int_M d\omega$, except for the last term (i.e. $i = n$ term), where instead of 0 we will get

$$\int_{\mathbb{R}^{n-1}} \left(\int_0^{\infty} \frac{\partial f_n}{\partial x^n} dx^n \right) dx^1 \cdots dx^{n-1} = - \int_{\mathbb{R}^{n-1}} f_n(x^1, \cdots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.$$

On the other hand, since $x^n = 0$ on ∂M, we see

$$\iota_{\partial M}^* \omega = (-1)^{n-1} f_n(x^1, \cdots, x^{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1} = -f_n(x^1, \cdots, x^{n-1}, 0) \cdot (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}.$$

So

$$\int_{\partial M} \omega = \int_{\mathbb{R}^{n-1}} (-f_n(x^1, \cdots, x^{n-1}, 0)) dx^1 \cdots dx^{n-1} = - \int_{\mathbb{R}^{n-1}} f_n(x^1, \cdots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.$$

Case (3): In general we cover the set $\mathrm{supp}(\omega)$ (which, by our assumption, is compact) by finitely many coordinate charts, and take a partition of unity as usual. Then

$$\int_{\partial M} \iota_{\partial M}^* \omega = \sum_i \int_{\partial M} \iota_{\partial M}^* (\rho_i \omega) = \sum_i \int_M d(\rho_i \omega) = \sum_i \int_M d\rho_i \wedge \omega + \int_M d\omega.$$

Now the conclusion follows from the fact

$$\sum_i \int_M d\rho_i \wedge \omega = \int_M d(\sum \rho_i) \wedge \omega = 0.$$

$\square$

Finally we give a couple applications.

Application I: Variation of volume.

Let M be a smooth manifold (without boundary), $\Omega \subset M$ be a domain so that $\overline{\Omega}$ is compact. Let μ be a volume form on M. Then we can define the *volume* of Ω with respect to μ to be

$$\mathrm{Vol}(\Omega) = \int_\Omega \mu.$$

Now assume that the boundary $\partial\Omega$ is a smooth submanifold of M, and let X be any complete vector field on M. Then X generates a flow, i.e. a family of diffeomorphisms $\phi_t : M \to M$. Note that these ϕ_t's are orientation-preserving. (Reason: $\det(d\phi_t) \neq 0$ for all t, and $\det(d\phi_t)$ is continuous with respect to t, and $\det(d\phi_0) \equiv 1$.)

Let $\Omega_t = \phi_t(\Omega)$ be the "flow-out" of Ω under this flow. We have

Proposition 2.5.

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(\Omega_t) = \int_{\partial\Omega} \iota_X \mu.$$

Proof. By definitions and the change of variable formula (note: ϕ_t's preserve the orientation!)

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(\Omega_t) = \left.\frac{d}{dt}\right|_{t=0} \int_{\Omega_t} \mu = \left.\frac{d}{dt}\right|_{t=0} \int_\Omega \phi_t^*\mu = \int_\Omega \left.\frac{d}{dt}\right|_{t=0} \phi_t^*\mu = \int_\Omega \mathcal{L}_X\mu.$$

Now apply Cartan's magic formula (PSet 6), we get (note: $d\mu = 0$ since it is an $(n+1)$-form)

$$\mathcal{L}_X\mu = d\iota_X\mu + \iota_X d\mu = d\iota_X\mu.$$

So by Stokes' formula,

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(\Omega_t) = \int_\Omega d\iota_X\mu = \int_{\partial\Omega} \iota_X\mu.$$

□

A special case: Let $M = \mathbb{R}^n$ (or more generally a Riemannian manifold). Let f be a smooth function, and 0 is a regular value of f. Let Ω be the sub-level set $\Omega = \{p : f(p) \leq 0\}$. Take the vector field X to be the gradient vector field ∇f of f. Then $\iota_X\mu$ is the induced volume form on the boundary $\partial\Omega$. In this setting, the above formula can be interpreted as "the rate of change of the sub-level sets of a smooth function along the gradient flow equals the surface area of the level set".

Application II: The divergence.

Let M be a smooth manifold of dimension n, μ a volume form on M, and X a smooth vector field on M. Then the Lie derivative $\mathcal{L}_X\mu$ is again a smooth n-form on M. So there is a unique smooth function $\mathrm{div}_\mu(X)$ on M so that

$$\mathcal{L}_X\mu = \mathrm{div}_\mu(X)\mu.$$

Definition 2.6. The function $\mathrm{div}_\mu(X)$ is called the *divergence* of X with respect to the volume form μ.

Remark. Suppose locally $\mu = f dx^1 \wedge \cdots \wedge dx^n$, and $X = \sum X^i \partial_i$. Then we have

$$\mathrm{div}_\mu X = \sum \frac{1}{f}\partial_i(fX^i).$$

As a consequence, for any $g \in C^\infty(M)$, we have

$$\mathrm{div}_\mu(gX) = g\mathrm{div}_\mu(X) + X(g)\mu.$$

The proofs are left as exercises.

In particular, if we take $M = \mathbb{R}^n$ and $\mu = dx^1 \wedge \cdots \wedge dx^n$, then

$$\mathrm{div}X = \sum_i \frac{\partial X^i}{\partial x^i},$$

which is the divergence of a vector field that we learned in calculus.

We have the following generalization of Gauss Theorem in calculus:

Proposition 2.7 (Gauss Theorem)**.**

$$\int_M (\mathrm{div}_\mu X)\mu = \int_{\partial M} \iota_X \mu.$$

Proof. By Cartan's magic formula and Stokes' formula,

$$\int_M (\mathrm{div}_\mu X)\mu = \int_M \mathcal{L}_X \mu = \int_M d\iota_X\mu = \int_{\partial M} \iota_X \mu.$$

□

A vector field X is called *divergence free* (or *incompressible*) if

$$\mathrm{div}_\mu(X) = 0.$$

If X is divergence free, then

$$\mathrm{div}_\mu(fg\mu) = X(fg)\mu = X(f)g\mu + fX(g)\mu.$$

It follows

Corollary 2.8. *If M is a compact manifold without boundary (i.e. $\partial M = \emptyset$), and X is a divergence free vector field on M, then for any $f, g \in C^\infty(M)$, one has*

$$\int_M X(f)g\mu = -\int_M fX(g)\mu.$$

(In other words, the vector field X, as an operator $X : C^\infty(M) \to C^\infty(M)$, is skew-symmetric.)