LECTURE 24: THE DE RHAM COHOMOLOGY

1. The De Rham cohomology

Let M be a smooth manifold. As we have seen, $d : \Omega^k(M) \to \Omega^{k+1}(\mathrm{M})$ is a linear map so that for any k and any $\omega \in \Omega^k(M)$,

$$d^2\omega = d(d\omega) = 0.$$

So we get a *de Rham cochain complex*

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

(Such a sequence of vector spaces [could be an infinite sequence] connected by a sequence of linear maps, such that the composition of any two consecutive maps is the zero map, is called a *chain complex* or a *cochain complex*, depending on the "direction" of the maps.)

Definition 1.1. Let M be a smooth manifold, and $\omega \in \Omega^k(M)$ is a k-form.

1. We say ω is *closed* if $d\omega = 0$.
2. We say ω is *exact* if there exists a $(k-1)$-form $\eta \in \Omega^{k-1}(M)$ such that $\omega = d\eta$.

We will denote the set of closed k-forms by $Z^k(M)$, and the set of exact k-forms by $B^k(M)$:

$$Z^k(M) = \text{the set of closed } k\text{-forms} = \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)),$$
$$B^k(M) = \text{the set of exact } k\text{-forms} = \mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M)).$$

Remark. Suppose $\dim M = n$. Then by definition we have

- For $k > n$: $B^k(M) = Z^k(M) = \{0\}$.
- For $k = 0$: $B^0(M) = \{0\}$, and
  $$Z^0(M) = \{f \in C^\infty(M) \mid df = 0\} \simeq \mathbb{R}^K.$$
  where K is the number of connected components of M.
- For $k = n$: $Z^n(M) = \Omega^n(M)$.

Since $d^2 = 0$, We have the following inclusion relation as vector spaces (and thus as additive groups)

$$B^k(M) \subset Z^k(M) \subset \Omega^k(M).$$

Definition 1.2. The quotient group

$$H^k_{dR}(M) := Z^k(M)/B^k(M)$$

is called the k^{th} *de Rham cohomology group* of M.

Given any $\omega \in Z^k(M)$, we will denote by $[\omega]$ the corresponding *cohomology class.*

Remark. Suppose $\dim M = n$. According to the above remark, we get

$$H^k_{dR}(M) = \{0\}, \qquad \forall k > n$$

and (we still denote by K the number of connected components of M)

$$H^0_{dR}(M) \simeq \mathbb{R}^K.$$

By definition, $H^k_{dR}(M)$ is a vector space. We will see that for many smooth manifolds (including all compact manifolds), $\dim H^k_{dR}(M) < \infty$ for all k. On the other hand, we have seen $\dim H^0(\mathbb{Z}) = \infty$. (As another example: it is not hard to prove $\dim H^1_{dR}(\mathbb{R}^2 \setminus \mathbb{Z}^2) = +\infty$.)

Definition 1.3. In the case $\dim H^k_{dR}(M) < \infty$, we will call the number

$$b_k(M) = \dim H^k_{dR}(M)$$

the k^{th} *Betti number* of M, and the number

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b_k(M)$$

the *Euler characteristic* of M.

Example. For $M = \mathbb{R}$, we have $H^0_{dR}(\mathbb{R}) \simeq \mathbb{R}$ and $H^k_{dR}(\mathbb{R}) = \{0\}$ for $k \geq 2$.

To calculate $H^1_{dR}(\mathbb{R})$, we just notice that any 1-form $g(t)dt$ on $\mathbb{R}$ is exact, since

$$\omega = g(t)dt \Longleftrightarrow \omega = dG, \text{where } G(t) = \int_0^t g(\tau)d\tau.$$

It follows that $Z^1(\mathbb{R}) = \Omega^1(\mathbb{R}) = B^1(\mathbb{R})$. So $H^1_{dR}(\mathbb{R}) = \{0\}$.

Example. Consider $M = S^1$. As we have seen,

$$H^0_{dR}(S^1) \simeq \mathbb{R} \qquad \text{and} \qquad H^k_{dR}(S^1) = 0 \quad \text{for} \ \ k \geq 2.$$

To calculate $H^1_{dR}(S^1)$, we argue as in the previous example. Note that on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, the "angle" variable θ is not a globally defined smooth function on S^1, but the translation invariance of d on $\mathbb{R}$ implies that the differential form $d\theta$ is a globally defined 1-form on S^1. (As a consequence, the 1-form $d\theta$ is a closed 1-form on S^1, but is not an exact 1-form on S^1. However, it is an exact 1-form on any proper subset of S^1.) So

$$Z^1(S^1) = \Omega^1(S^1) = \{f d\theta \mid f \in C^\infty(S^1)\} \simeq \{f \in C^\infty(\mathbb{R}) \mid f \text{ is periodic with period } 2\pi\}.$$

On the other hand, by the fundamental theorem of calculus,

$$\begin{aligned} \omega \text{ is exact} &\Longleftrightarrow \omega = df, \text{ where } f \text{ is periodic with period } 2\pi \\ &\Longleftrightarrow \omega = g(\theta)d\theta, \text{ where } g \text{ is periodic with period } 2\pi \text{ and } \int_0^{2\pi} g(\theta)d\theta = 0. \end{aligned}$$

So we conclude

$$H^1_{dR}(S^1) \simeq \frac{\{f \in C^\infty(\mathbb{R}) \mid f \text{ is periodic with period } 2\pi\}}{\{g \in C^\infty(\mathbb{R}) \mid g \text{ is periodic with period } 2\pi, \text{ and } \int_0^{2\pi} g(\theta)d\theta = 0\}}.$$

This implies that

$$H^1_{dR}(S^1) \simeq \mathbb{R},$$

since the linear map

$$\varphi : H^1_{dR}(S^1) \to \mathbb{R}, \qquad [f] \mapsto \int_0^{2\pi} f(\theta) d\theta.$$

is an linear isomorphism:

- φ is well-defined: $[f_1] = [f] \implies f_1 - f \in B^1(S^1) \implies \int_0^{2\pi} f_1(\theta)d\theta = \int_0^{2\pi} f(\theta)d\theta$.
- φ is injective: $[f_1] \neq [f] \implies f_1 - f \notin B^1(S^1) \implies \int_0^{2\pi} f_1(\theta)d\theta \neq \int_0^{2\pi} f(\theta)d\theta$.
- φ is surjective: for any $c \in \mathbb{R}$, $f(\theta) := c \in Z^1(S^1) \implies \varphi([f]) = \int_0^{2\pi} f(\theta)d\theta = 2\pi c$.

One can extend the wedge product and pull-back operations on differential forms to operations on cohomology classes.

First let $\omega \in Z^k(M)$ and $\eta \in Z^l(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = 0,$$

i.e. $\omega \wedge \eta \in Z^{k+l}(M)$. Moreover, for any $\xi_1 \in \Omega^{k-1}(M)$ and $\xi_2 \in \Omega^{l-1}(M)$,

$$(\omega + d\xi_1) \wedge (\eta + d\xi_2) = \omega \wedge \eta + d\left[(-1)^k \omega \wedge \xi_2 + (-1)^{k-1}\xi_1 \wedge \eta + (-1)^{k-1}\xi_1 \wedge d\xi_2\right].$$

In other words, $[\omega \wedge \eta]$ is independent of the choice of ω and η in $[\omega]$ and $[\eta]$. So we can define

Definition 1.4. The *cup product* between $[\omega] \in H^k_{dR}(M)$ and $[\eta] \in H^l_{dR}(M)$ is

$$[\omega] \cup [\eta] := [\omega \wedge \eta] \in H^{k+l}_{dR}(M).$$

Similarly suppose $\varphi : M \to N$ is smooth. Then the fact $d\varphi^* = \varphi^* d$ implies

$$\varphi^*(Z^k(N)) \subset Z^k(M) \quad \text{and} \quad \varphi^*(B^k(N)) \subset B^k(M).$$

It follows that $\varphi^* : \Omega^k(N) \to \Omega^k(M)$ descends to a pull-back $\varphi^* : H^k_{dR}(N) \to H^k_{dR}(M)$:

$$\varphi^*([\omega]) := [\varphi^* \omega].$$

Obviously φ^* is a group homomorphism. It is easy to check

- $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
- $\mathrm{Id}^* = \mathrm{Id}$.

As an immediate consequence, we see that the de Rham cohomology is a smooth invariant:

Corollary 1.5 (Diffeomorphism Invariance). *If $\varphi : M \to N$ is a diffeomorphism, then*

$$\varphi^* : H^k_{dR}(N) \to H^k_{dR}(M)$$

is a linear isomorphism for all k. In particular, $b_k(N) = b_k(M)$ for all k, and $\chi(N) = \chi(M)$.

Remark. For any smooth map $\varphi : M \to N$, The cup product makes

$$H^*_{dR}(M) = \bigoplus_{k=0}^{n} H^k_{dR}(M)$$

a graded ring, and the induced map φ^* is in fact a ring homomorphism

$$\varphi^* : H^*_{dR}(N) \to H^*_{dR}(M),$$

since

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta.$$

Moreover, if φ is a diffeomorphism, then $\varphi^* : H^*_{dR}(N) \to H^*_{dR}(M)$ is a ring isomorphism.

2. HOMOTOPIC INVARIANCE

In this section we shall prove a much stronger result: if two manifolds are homotopy equivalent, then they have the same de Rham cohomology groups. We first give the definition:

Definition 2.1. Two smooth manifolds (it is enough to assume that they are topological spaces) M and N are said to be *homotopy equivalent* if there exist continuous maps $\varphi : M \to N$ and $\psi : N \to M$ so that $\varphi \circ \psi$ is homotopic to Id_N and $\psi \circ \varphi$ is homotopic to Id_M.

Recall: Two continuous maps $f_0 : M \to N$ and $f_1 : M \to N$ are *homotopic* if there is a continuous map $F : M \times \mathbb{R} \to N$ so that for all $p \in M$,

$$F(p, 0) = f_0(p) \qquad \text{and} \qquad F(p, 1) = f_1(p).$$

If f_0 and f_1 are homotopic, we denote $f_0 \sim f_1$. In Lecture 10 we have shown

- any continuous map is homotopic to some smooth map.
- any two homotopic smooth maps are smoothly homotopic (i.e., the homotopy F can be chosen to be smooth if both f_0 and f_1 are smooth).

Homotopy equivalence is an equivalence relation which is much weaker than homeomorphism or diffeomorphism. For example, manifolds of different dimensions could be homotopy equivalent.

Example. S^{n-1} is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$ via maps $\iota : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ and

$$\psi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}, \quad x \mapsto \frac{x}{|x|}.$$

Clearly $\psi \circ \iota = \mathrm{Id}_{S^{n-1}}$. The map $\iota \circ \psi(x) = \frac{x}{|x|}$ is homotopic to $\mathrm{Id}_{\mathbb{R}^n \setminus \{0\}}$ via

$$\Phi(x, t) = tx + (1 - t)\frac{x}{|x|}.$$

Example. A *star-shaped region* in $\mathbb{R}^n$ is a region $U \subset \mathbb{R}^n$ satisfying the following property:

$$\exists x_0 \in U \text{ so that } \forall x \in U \text{ and } \forall 0 \le t \le 1, \text{ one has } tx_0 + (1 - t)x \in U.$$

Obviously a star-shaped region is homotopic equivalent to a single point set $\{x_0\}$. (What are the maps that give the homotopy equivalence?)

Example. Any topological space that is homotopic to a single point set is called *contractible*. Obviously $\mathbb{R}^n$ and any star-like region is contractible. It is easy to prove: if N is contractible, then for any smooth manifold M, the product $M \times N$ is homotopic to M.

We would like to prove

Theorem 2.2 (Homotopy Invariance). *If M and N are homotopy equivalent, then*

$$H^k_{dR}(M) \simeq H^k_{dR}(N), \qquad \forall k.$$

Obviously the homotopy invariance implies the topological Invariance of cohomology groups:

If M is homeomorphic to N, then $H^k_{dR}(M) \simeq H^k_{dR}(N)$ for all k.

Remark. Although in defining $H^k_{dR}(M)$, we need to use the smooth structure on M (to define the operator d and the space $\Omega^k(M)$ etc), the topological invariance tells us that $H^k_{dR}(M)$ only depends on the topology of M, and is independent of the smooth structure! In fact, for any topological space X, one can define its *singular cohomology groups* $H^k_{sing}(X, \mathbb{R})$ which depends only on the topology (and in fact only depends on the homotopy class) of X. The famous theorem of de Rham claims

Theorem 2.3 (The de Rham theorem). $H^k_{dR}(M) = H^k_{sing}(M, \mathbb{R})$ *for all k.*

We will not prove the theorem in this course.

Another immediate consequence of the homotopy invariance is

Corollary 2.4 (Poincare's lemma). *If U is a star-shaped region in $\mathbb{R}^n$, then for any $k \geq 1$, $H^k_{dR}(U) = 0$. In particular, $H^k_{dR}(\mathbb{R}^n) = 0$ for all $k \geq 1$.*

Since any point in a manifold has a neighborhood that is homeomorphic to a star-like region in $\mathbb{R}^n$, we conclude that any closed form is locally exact:

Corollary 2.5. *Suppose $k \geq 1$. Then for any closed k-form $\omega \in Z^k(M)$ and any $p \in M$, there is a neighborhood U of p and an $(k-1)$-form $\eta \in \Omega^{k-1}(U)$ so that $\omega = d\eta$ on U.*

The homotopy invariance is a consequence of

Theorem 2.6. *Let $f, g \in C^\infty(M, N)$ be homotopic, then $f^* = g^* : H^k_{dR}(N) \to H^k_{dR}(M)$.*

Proof of Theorem 2.2. Let $\varphi : M \to N$ and $\psi : N \to M$ be continuous maps so that $\varphi \circ \psi \sim \mathrm{Id}_N$ and $\psi \circ \varphi \sim \mathrm{Id}_M$. Then one can find $\varphi_1 \in C^\infty(M, N)$ and $\psi_1 \in C^\infty(N, M)$ so that $\varphi_1 \sim \varphi$ and $\psi_1 \sim \psi$. It follows that both $\varphi_1 \circ \psi_1$ and $\psi_1 \circ \varphi_1$ are smooth, and $\varphi_1 \circ \psi_1 \sim \mathrm{Id}_N$, $\psi_1 \circ \varphi_1 \sim \mathrm{Id}_M$.

Applying Theorem 2.6, we get

$$\varphi_1^* \circ \psi_1^* = \mathrm{Id} : H^k_{dR}(M) \to H^k_{dR}(M)$$
$$\psi_1^* \circ \varphi_1^* = \mathrm{Id} : H^k_{dR}(N) \to H^k_{dR}(N).$$

So φ^* and ψ^* are linear isomorphisms [and are ring isomorphisms from $H^*_{dR}(N)$ to $H^*_{dR}(M)$]. □

As in Algebraic Topology, Theorem 2.6 can be proved by constructing a cochain homotopy:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega^{k-1}(N) \arrow[r, "d"] \arrow[dl, dashed, "h"'] \arrow[d, shift right, "g^*"'] \arrow[d, shift left, "f^*"] & \Omega^{k}(N) \arrow[r, "d"] \arrow[dl, dashed, "h"'] \arrow[d, shift right, "g^*"'] \arrow[d, shift left, "f^*"] & \Omega^{k+1}(N) \arrow[r, "d"] \arrow[dl, dashed, "h"'] \arrow[d, shift right, "g^*"'] \arrow[d, shift left, "f^*"] & \cdots \arrow[dl, dashed, "h"'] \\
\cdots \arrow[r, "d"] & \Omega^{k-1}(M) \arrow[r, "d"] & \Omega^{k}(M) \arrow[r, "d"] & \Omega^{k+1}(M) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

The cochain homotopy between f^* and g^* is a sequence of maps $h_k : \Omega^k(N) \to \Omega^{k-1}(M)$ satisfying

$$g^* - f^* = d_M h_k + h_{k+1} d_N$$

on $\Omega^k(N)$.

Proof of Theorem 2.6 assuming the existence of the cochain homotopy.

Suppose $[\omega] \in H^k_{dR}(N)$. Then $d\omega = 0$ since ω is closed. It follows

$$g^*\omega - f^*\omega = (dh + hd)\omega = dh\omega \in B^k(M)$$

Thus $f^*([\omega]) = [f^*\omega] = [g^*\omega] = g^*([\omega])$. □

It remains to construct the cochain homotopy. We will use the flow generated by a vector field to complete the construction. Recall that if X is a complete vector field on M, then X generates a flow $\phi_t : M \to M$. We need

Lemma 2.7. *Let X be a complete vector field on M, and ϕ_t the flow generated by X. Then there is a linear operator $Q : \Omega^k(M) \to \Omega^{k-1}(M)$ so that for any $\omega \in \Omega^k(M)$,*

$$\phi_1^*\omega - \omega = dQ(\omega) + Q(d\omega).$$

Proof. Left as an exercise. Hint: $\phi_1^*\omega - \omega = \int_0^1 \left(\frac{d}{dt}\phi_t^*\omega\right) dt$, then apply Cartan's magic formula. □

Construction of the cochain homotopy $h_k : \Omega^k(N) \to \Omega^{k-1}(M)$.

Let $W = M \times \mathbb{R}$, then $X = \frac{\partial}{\partial t}$ is a complete vector field on W whose flow is

$$\phi_t(p, a) = (p, a + t).$$

By Lemma 2.7, there is a linear operator $Q : \Omega^k(W) \to \Omega^{k-1}(W)$ so that

$$\phi_1^*\omega - \omega = dQ(\omega) + Q(d\omega).$$

We let $\iota : M \to W$ be the map $\iota(p) = (p, 0)$, then

$$f = F \circ \iota \quad \text{and} \quad g = F \circ \phi_1 \circ \iota,$$

where $F : W \to N$ is a smooth homotopy between f and g. It follows that for any $\omega \in \Omega^k(N)$,

$$g^*\omega - f^*\omega = \iota^*\phi_1^*F^*\omega - \iota^*F^*\omega = \iota^*(dQ + Qd)F^*\omega = (d\iota^*QF^* + \iota^*QF^*d)\omega.$$

So if we denote $h = \iota^*QF^*$, then $h : \Omega^k(N) \to \Omega^{k-1}(M)$ satisfies

$$g^*\omega - f^*\omega = (dh + hd)\omega,$$

i.e. h is the cochain homotopy we are looking for. □