LECTURE 25: THE MAYER-VIETORIS SEQUENCE

1. The Mayer-Vietoris sequence

Suppose we have a *cochain complex* $(\mathcal{A}, d)$, i.e. a sequence

$$\cdots \xrightarrow{d_{k-2}} A^{k-1} \xrightarrow{d_{k-1}} A^k \xrightarrow{d_k} A^{k+1} \xrightarrow{d_{k+1}} A^{k+2} \xrightarrow{d_{k+2}} \cdots$$

where A^k's are vector spaces, and d_k's are linear maps such that

$$d_k \circ d_{k-1} = 0, \qquad \forall k.$$

Obviously we have $\mathrm{Im}(d_k) \subset \mathrm{Ker}(d_{k+1})$. One can define cohomology groups for any cochain complex as we did for the de Rham cochain complex:

$$H^k(\mathcal{A}) := \ker(d_k)/\mathrm{Im}(d_{k-1})$$

Such a complex $(\mathcal{A}, d)$ is called an *exact sequence* if $H^k(\mathcal{A}, d) = 0$ for all k, i.e.

$$\mathrm{Im}(d_{k-1}) = \ker(d_k), \qquad \forall k.$$

Note that if an exact sequence starts with 0,

$$(\cdots \to) \quad 0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \xrightarrow{d_3} \cdots,$$

then $d_1 : V^1 \to V^2$ is injective, while if an exact sequence ends with 0,

$$\cdots \xrightarrow{d_{k-2}} V^{k-1} \xrightarrow{d_{k-1}} V^k \xrightarrow{d_k} V^{k+1} \xrightarrow{d_{k+1}} 0 \quad (\to \cdots),$$

then $d_k : V^k \to V^{k+1}$ is surjective. In particular, if we have a *short exact sequence*

$$0 \longrightarrow V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \longrightarrow 0,$$

then d_1 is injective, d_2 is surjective, and

$$V^2 \simeq \ker(d_2) \oplus \mathrm{Im}(d_2) \simeq V^1 \oplus V^3.$$

Another useful fact about exact sequences is: if a finite sequence

$$0 \to A^1 \to A^2 \to \cdots \to A^k \to 0$$

is exact, then $\sum_i (-1)^i \dim A^i = 0$. (The proof is left as an exercise.)

A general principle in homological algebra is: if we have three cochain complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$, which form a short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ in the sense that for any k,

$$0 \to A^k \to B^k \to C^k \to 0$$

is a short exact sequence, then one can construct a long exact sequence of the cohomology groups

$$\cdots \to H^{k-1}(\mathcal{C}) \to H^k(\mathcal{A}) \to H^k(\mathcal{B}) \to H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A}) \to \cdots.$$

Now suppose M is a smooth manifold, and U, V are open sets in M so that $M = U \cup V$. Since M, U, V and $U \cap V$ are all smooth manifolds, we have four de Rham complexes,

$$\begin{aligned}
\Omega^*(M):& \qquad 0 \to \Omega^0(M) \to \Omega^1(M) \to \Omega^2(M) \to \Omega^3(M) \to \cdots \\
\Omega^*(U):& \qquad 0 \to \Omega^0(U) \to \Omega^1(U) \to \Omega^2(U) \to \Omega^3(U) \to \cdots \\
\Omega^*(V):& \qquad 0 \to \Omega^0(V) \to \Omega^1(V) \to \Omega^2(V) \to \Omega^3(V) \to \cdots
\end{aligned}$$

and

$$\Omega^*(U \cap V): \qquad 0 \to \Omega^0(U \cap V) \to \Omega^1(U \cap V) \to \Omega^2(U \cap V) \to \Omega^3(U \cap V) \to \cdots$$

It turns out that these complexes form a short exact sequence

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0.$$

To see this, we consider the inclusion maps

$$\iota_1 : U \hookrightarrow M, \quad \iota_2 : V \hookrightarrow M$$

and inclusion maps

$$\jmath_1 : U \cap V \hookrightarrow U, \quad \jmath_2 : U \cap V \hookrightarrow V.$$

These inclusion maps induce linear maps between the spaces of k-forms (and also induce linear maps between the corresponding de Rham cohomology groups, which we use the same notation)

$$\alpha_k : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V), \quad \omega \mapsto (\iota_1^*\omega, \iota_2^*\omega)$$

and

$$\beta_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V), \quad (\omega_1, \omega_2) \mapsto \jmath_1^*\omega_1 - \jmath_2^*\omega_2.$$

The short exact sequence we alluded above is stated in the following proposition. The proof is simple and is left as an exercise.

Proposition 1.1. *For any k, the sequence*

$$0 \longrightarrow \Omega^k(M) \xrightarrow{\alpha_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta_k} \Omega^k(U \cap V) \longrightarrow 0$$

is a short exact sequence.

According to the general principle in homological algebra that we mentioned above, we should be able to construct a long exact sequence consisting of corresponding de Rham cohomology groups. Let's explain the construction in our setting in detail.

The maps α_k and β_k defined above induces maps

$$\alpha_k : H^k_{dR}(M) \to H^k_{dR}(U) \oplus H^k_{dR}(V)$$

and

$$\beta_k : H^k_{dR}(U) \oplus H^k_{dR}(V) \to H^k_{dR}(U \cap V).$$

To get a long exact sequence, we need to define a linear map (called the *connecting homomorphism*)

$$\delta_k : H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(M).$$

The map δ_k can be defined by the standard "diagram chasing" method. In what follows we will give an explicit construction: We fix a partition of unity $\{\rho_U, \rho_V\}$ subordinate to the cover $\{U, V\}$ of M. For any $\omega \in Z^k(U \cap V)$, we define

$$\delta_k([\omega]) := [\eta],$$

where η is the $(k+1)$-form

$$\eta = \begin{cases} d(\rho_V \omega) & \text{on } U, \\ -d(\rho_U \omega) & \text{on } V. \end{cases}$$

(Why should we define η like this? Well, we need a $(k+1)$-form on M. The only way to get such a smooth $(k+1)$-form on M is to take the exterior derivative of a smooth k-form on M. Unfortunately the zero extension of ω to M is not a smooth k-form on M since ω could be "nonzero near the boundary of $U \cap V$. P.O.U is the correct way to "force ω to be zero near the boundaries"!)

Lemma 1.2. *The map δ_k is well-defined.*

Proof. There are many issues to be checked:

- $\rho_V \omega \in \Omega^k(U)$: since $\rho_V \in C^\infty(M)$ and $\operatorname{supp}(\rho_V) \cap U \subset U \cap V$. (Note: (1) ω is NOT defined on $U \setminus U \cap V$. (2) $\rho_V \omega \notin \Omega^k(M)$.) Similarly $\rho_U \omega \in \Omega^k(V)$.
- $\eta \in \Omega^{k+1}(M)$: Since $\rho_U + \rho_V = 1$ and since $d\omega = 0$ on $U \cap V$, we see
$$d(\rho_V \omega) = -d(\rho_U \omega)$$
on $U \cap V$, so η is a well-defined smooth $(k+1)$-form on M.
- $\eta \in Z^{k+1}(M)$: By definition we have $d\eta = 0$ on both U and V. So $\eta \in Z^{k+1}(M)$.
- $[\eta]$ is independent of the choices of ρ_U and ρ_V: Let $\tilde{\rho}_U$ and $\tilde{\rho}_V$ be another partition of unity subordinate to the cover $\{U, V\}$, and let $\tilde{\eta}$ be the resulting $(k+1)$-form. Then $\tilde{\rho}_V - \rho_V = \rho_U - \tilde{\rho}_U$ is supported in $U \cap V$. So if we let
$$\xi = (\tilde{\rho}_V - \rho_V)\omega,$$
then it is a smooth k-form on M, and by the construction, we have $\tilde{\eta} - \eta = d\xi$ on both U and V, and thus on M.
- $[\eta]$ is independent of the choices of ω: Suppose $\tilde{\omega} = \omega + d\zeta$ and denote the resulting $(k+1)$-from by $\tilde{\eta}$. Then
$$\tilde{\eta} - \eta = \begin{cases} d(\rho_V d\zeta) & \text{on } U, \\ -d(\rho_U d\zeta) & \text{on } V. \end{cases}$$
We define
$$\xi = \begin{cases} -d\rho_V \wedge \zeta & \text{on } U, \\ d\rho_U \wedge \zeta & \text{on } V. \end{cases}$$
Then by the same argument above, ξ is a smooth k-form on M. Moreover,
$$d\xi = \left\{ \begin{array}{rll} d\rho_V \wedge d\zeta = & d(\rho_V d\zeta) & \text{on } U \\ -d\rho_U \wedge d\zeta = & -d(\rho_U d\zeta) & \text{on } V \end{array} \right\} = \tilde{\eta} - \eta.$$
So $[\tilde{\eta}] = [\eta]$ and the conclusion follows.

□

Now we can state the main theorem:

Theorem 1.3 (Mayer-Vietories). *Let U, V be open sets in M so that $M = U \cup V$. Then we have a long exact sequence*

$$\cdots \xrightarrow{\delta_{k-1}} H^k_{dR}(M) \xrightarrow{\alpha_k} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{\beta_k} H^k_{dR}(U \cap V) \xrightarrow{\delta_k} H^{k+1}_{dR}(M) \xrightarrow{\alpha_{k+1}} \cdots$$

Proof. One has to show

- $\mathrm{Im}(\alpha_k) = \ker(\beta_k)$,
- $\mathrm{Im}(\beta_k) = \ker(\delta_k)$,
- $\mathrm{Im}(\delta_k) = \ker(\alpha_{k+1})$.

This amounts to prove 6 inclusion relations. We will prove one of them and leave the rest as exercises.

$\boxed{\text{Proof of } \mathrm{Im}(\beta_k) \subset \ker(\delta_k)}$: Suppose $\omega_1 \in Z^k(U)$, $\omega_2 \in Z^k(V)$. Let

$$\omega := \beta_k(\omega_1, \omega_2) = \jmath_1^*\omega_1 - \jmath_2^*\omega_2 \in \Omega^k(U \cap V).$$

Then $\delta_k([\omega]) = [\eta]$, where

$$\eta = \begin{cases} d(\rho_V\omega) = d(\rho_V\omega - \omega_1) & \text{on } U, \\ -d(\rho_U\omega) = -d(\rho_U\omega + \omega_2) & \text{on } V. \end{cases}$$

Note that on $U \cap V$,

$$\rho_V\omega - \jmath_1^*\omega_1 = -\rho_U\omega - \jmath_2^*\omega_2.$$

So there is a smooth k-form ξ on M so that

$$\xi = \begin{cases} \rho_V\omega - \omega_1 & \text{on } U, \\ -\rho_U\omega - \omega_2 & \text{on } V. \end{cases}$$

As a consequence, $\eta = d\xi$ and thus $[\eta] = 0$. ☐

2. Applications of Mayer-Vietoris sequence

The Mayer-Vietories sequence is a very powerful tool which has many applications.

$\boxed{\text{Application 1: The de Rham cohomology groups of the spheres}}$

Theorem 2.1. *For $n \ge 1$, $H^k_{dR}(S^n) \simeq \begin{cases} \mathbb{R}, & k = 0, n, \\ 0, & 1 \le k \le n-1. \end{cases}$.*

Proof. Last time we have proven

$$H^0_{dR}(S^n) \simeq \mathbb{R} \quad \text{and} \quad H^1_{dR}(S^1) \simeq \mathbb{R}.$$

In what follows we will prove

1. For $n \ge 2$, $H^1_{dR}(S^n) = 0$.
2. For $n \ge 2, k \ge 2$, $H^k_{dR}(S^n) \simeq H^{k-1}_{dR}(S^{n-1})$.

Obvious these results together imply the theorem.

For $n \ge 2$, we let

$$U = S^n - \{(0, \cdots, 0, -1)\} \quad \text{and} \quad V = S^n - \{(0, \cdots, 0, 1)\}.$$

Then

- $M = U \cup V$,
- U and V are diffeomorphic to $\mathbb{R}^n$,
- $U \cap V$ is homotopy equivalent to S^{n-1}.

To prove (1), we look at the beginning of the Mayer-Vietories sequence

$$0 \longrightarrow H^0_{dR}(S^n) \longrightarrow H^0_{dR}(U) \oplus H^0_{dR}(V) \longrightarrow H^0_{dR}(U \cap V) \longrightarrow H^1(S^n) \to H^1_{dR}(U) \oplus H^1_{dR}(V),$$

which now becomes

$$0 \longrightarrow \mathbb{R} \xrightarrow{\alpha_0} \mathbb{R}^2 \xrightarrow{\beta_0} \mathbb{R} \xrightarrow{\delta_0} H^1_{dR}(S^n) \to 0.$$

Since α_0 is injective,

$$\dim \ker(\beta_0) = \dim \mathrm{Im}(\alpha_0) = 1.$$

It follows that

$$\dim \mathrm{Im}(\beta_0) = \dim \mathbb{R}^2 - \dim \ker(\beta_0) = 1,$$

i.e. β_0 is surjective. So $\ker(\delta_0) = \mathbb{R}$, i.e. $\delta_0 \equiv 0$. But by exactness, δ_0 is surjective. This implies $H^1_{dR}(S^n) = 0$.

To prove (2), we look at the following part of the Mayer-Vietories sequence

$$H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) \xrightarrow{\beta_{k-1}} H^{k-1}_{dR}(U \cap V) \xrightarrow{\delta_{k-1}} H^k_{dR}(S^n) \xrightarrow{\alpha_k} H^k_{dR}(U) \oplus H^k_{dR}(V),$$

which becomes

$$0 \xrightarrow{\beta_{k-1}} H^{k-1}_{dR}(S^{n-1}) \xrightarrow{\delta_{k-1}} H^k_{dR}(S^n) \xrightarrow{\alpha_k} 0.$$

By exactness, the map δ_{k-1} has to be both injective and surjective, and thus must be a linear isomorphism. This proves (2). □

As a consequence, we can prove the topological invariance of the dimension: (Recall: In Lecture 1 we proved this theorem for $n = 1$ by counting the number of connected components, which is really the 0th de Rham cohomology group H^0_{dR}!)

Corollary 2.2 (Topological Invariance of Dimension). *If $m \neq n$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.*

Proof. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism, then $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$ is a homeomorphism. So

$$H^k_{dR}(\mathbb{R}^n \setminus \{0\}) = H^k_{dR}(\mathbb{R}^m \setminus \{f(0)\}, \qquad \forall k.$$

But $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}, while $\mathbb{R}^m \setminus \{f(0)\}$ is homotopic to S^{m-1}. So

$$H^k_{dR}(S^{m-1}) = H^k_{dR}(S^{n-1}), \qquad \forall k.$$

This contradicts with the fact $m \neq n$. □

$\boxed{\text{Application 2: } \dim H^k_{dR}(M) < \infty \text{ for many smooth manifolds}}$

Definition 2.3. Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in\Lambda}$ an open cover of M. We say $\{U_\alpha\}_{\alpha\in\Lambda}$ is a *good cover* if for any finite subset $I = \{\alpha_1, \cdots, \alpha_k\} \subset \Lambda$ of indices, the intersection

$$U_I := U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_k}$$

is either empty or diffeomorphic to $\mathbb{R}^n$.

Example. The cover of S^n by $S^n - \{N\}$ and $S^n - \{S\}$ is not a good cover. However, the cover of S^n by the $2n+2$ hemispheres

$$S^n_{i,+(-)} = \{(x^1, \cdots, x^{n+1}) \mid x^i > (<) 0\}$$

is a good cover.

Using Riemannian geometry, one can show that any open cover of any smooth manifold M admits a refinement which is a good cover. (Use the so-called geodesically convex neighborhoods.) In particular, if M is compact, then M admits a good cover which contains only finitely many sets. Such a cover is called a *finite good cover.*

Obviously any sub-cover of a good cover is still a good cover.

Theorem 2.4. *If M admits a finite good cover, then $H^k_{dR}(M)$ is finite dimensional.*

Proof. We proceed by induction on the number of sets in a finite good cover of M. If M admits a good cover that contains only one open set, then that open set has to be M itself. In this case M is diffeomorphic to $\mathbb{R}^n$ and the conclusion follows.

Now suppose the theorem holds for any manifold that admits a good cover containing $l-1$ open sets. Let M be a manifold with an good cover $\{U_1, \cdots, U_l\}$. We denote

$$U = U_1 \cup \cdots \cup U_{l-1} \qquad \text{and} \qquad V = U_l.$$

Then $U \cap V$ admits a finite good cover $\{U_1 \cap U_l, \cdots, U_{l-1} \cap U_l\}$. By the induction hypothesis, all the de Rham cohomology groups of U, V and $U \cap V$ are finite dimensional. Now consider the Mayer-Vertoris sequence

$$\cdots \longrightarrow H^{k-1}_{dR}(U \cap V) \xrightarrow{\delta_{k-1}} H^k_{dR}(M) \xrightarrow{\alpha_k} H^k_{dR}(U) \oplus H^k_{dR}(V) \longrightarrow .$$

The conclusion follows since

$$\dim \operatorname{Im}(\alpha_k) \le \dim H^k_{dR}(U) \oplus H^k_{dR}(V) < \infty$$

and

$$\dim \ker(\alpha_k) = \dim \operatorname{Im}(\delta_{k-1}) \le \dim H^{k-1}_{dR}(U \cap V) < \infty.$$

$\square$

As a consequence, we immediately get

Corollary 2.5. *If M is compact or M is homotopy equivalent to a compact manifold, then $\dim H^k_{dR}(M) < \infty$ for all k.*

Application 3: The Kunneth formula

Theorem 2.6. *Let M and N be manifolds with finite good covers. Then for any $0 \leq k \leq \dim M + \dim N$, one has*

$$H^k_{dR}(M \times N) \simeq \bigoplus_{i=0}^{k} H^i_{dR}(M) \otimes H^{k-i}_{dR}(N).$$

Sketch of proof. Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N$ be the standard projections. Then we get a map

$$\Psi : \Omega^*(M) \otimes \Omega^*(N) \to \Omega^*(M \times N), \quad \omega_1 \otimes \omega_2 \mapsto \pi_M^*\omega_1 \wedge \pi_N^*\omega_2.$$

One can check that this map induces a map on cohomologies

$$\Psi : H^*_{dR}(M) \otimes H^*_{dR}(N) \to H^*_{dR}(M \times N), \quad [\omega_1] \otimes [\omega_2] \mapsto [\pi_M^*\omega_1 \wedge \pi_N^*\omega_2].$$

To prove that this Ψ is in fact an linear isomorphism, we do induction on the number l of elements in a good cover of M.

If $l = 1$, i.e. M is diffeomorphic to $\mathbb{R}^n$, then the Kunneth formula follows from the fact that $\mathbb{R}^n \times N$ is homotopy equivalent to N, and $H^k_{dR}(\mathbb{R}^n) = \mathbb{R}$ for $k = 0$ and $H^k_{dR}(\mathbb{R}^n) = 0$ for other k's.

Now suppose the Kunneth formula is proved for manifolds M admitting a good cover with no more than $l - 1$ open sets, and supppose now that $M = U_1 \cup \cdots \cup U_l$ is a good cover. Again we let $U = U_1 \cup \cdots U_{n-1}$ and $V = U_n$. For simplicity we will denote

$$\widetilde{H}^k(M, N) := \bigoplus_{i=0}^{k} H^i_{dR}(M) \otimes H^{k-i}_{dR}(N).$$

Consider the following diagram

$$\begin{tikzcd}
\widetilde{H}^k(M,N) \arrow[r, "\alpha"'] \arrow[d, "\Psi"'] & \widetilde{H}^k(U,N) \oplus \widetilde{H}^k(V,N) \arrow[r, "\beta"'] \arrow[d, "\Psi"'] & \widetilde{H}^k(U \cap V, N) \arrow[r, "\delta"'] \arrow[d, "\Psi"'] & \widetilde{H}^{k+1}(M,N) \arrow[d, "\Psi"'] \\
H^k_{dR}(M \times N) \arrow[r, "\alpha"'] & H^k_{dR}(U \times N) \oplus H^k_{dR}(V \times N) \arrow[r, "\beta"'] & H^k_{dR}(U \cap V \times N) \arrow[r, "\delta"'] & H^{k+1}_{dR}(M \times N)
\end{tikzcd}$$

where the horizontal maps α, β, δ's are the ones that is induced in the obvious way from the $\alpha_k, \beta_k, \delta_k$'s that we defined above. For this diagram we have

- By using the Mayer-Vietoris sequence one can prove that the two rows are exact.
- Moreover, one can prove that the diagram is commutative, i.e. we have $\Psi \circ \alpha = \alpha \circ \Psi$, $\Psi \circ \beta = \beta \circ \Psi$ and $\Psi \circ \delta = \delta \circ \Psi$. [The first two equalities are easy to prove, while the last one is much more complicated.]
- By using the induction hypothesis, the second and the third Ψ are linear isomorphisms.

Now the result follows from the following well-known Five lemma in homological algebra. $\square$

Lemma 2.7 (Five lemma). *Suppose we have the following commutative diagram*

$$\begin{tikzcd}
V_1 \arrow[r, "\alpha_1"'] \arrow[d, "\gamma_1"'] & V_2 \arrow[r, "\alpha_2"'] \arrow[d, "\gamma_2"'] & V_3 \arrow[r, "\alpha_3"'] \arrow[d, "\gamma_3"'] & V_4 \arrow[r, "\alpha_4"'] \arrow[d, "\gamma_4"'] & V_5 \arrow[d, "\gamma_5"'] \\
W_1 \arrow[r, "\beta_1"'] & W_2 \arrow[r, "\beta_2"'] & W_3 \arrow[r, "\beta_3"'] & W_4 \arrow[r, "\beta_4"'] & W_5
\end{tikzcd}$$

where each V_i is a linear space, and each map is a linear map. Suppose the two rows are exact, and the vertical maps $\gamma_1, \gamma_2, \gamma_4, \gamma_5$ are isomorphisms, then the map γ_3 is also an isomorphism.

Proof. This is a standard diagram-chasing exercise. $\square$

As a simple consequence of the Kunneth formula, we can compute

$$H^k_{dR}(S^n \times S^n) \simeq \begin{cases} \mathbb{R}, & k = 0 \text{ or } 2n, \\ \mathbb{R}^2, & k = n, \\ 0, & \text{other } k \end{cases}$$

and for $m \neq n$,

$$H^k_{dR}(S^m \times S^n) \simeq \begin{cases} \mathbb{R}, & k = 0, m, n \text{ or } m + n, \\ 0, & \text{other } k \end{cases}$$

We can also compute the de Rham cohomology groups of the n-torus $\mathbb{T}^n$.

Proposition 2.8. *For the n-dimensional torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$,*

$$H^k_{dR}(\mathbb{T}^n) \simeq \mathbb{R}^{\binom{n}{k}}.$$

The proof is left as an exercise.