LECTURE 26: COMPACTLY SUPPORTED DE RHAM COHOMOLOGY

1. Compactly Supported De Rham cohomology

For any $\omega \in M$, we can define the support of ω to be

$$\operatorname{supp}(\omega) = \overline{\{p \in M \mid \omega_p \neq 0\}}.$$

As usual, we say ω is *compactly supported* if $\operatorname{supp}(\omega)$ is compact in M. Let

$$\Omega_c^k(M) = \{\omega \in \Omega^k(M) \mid \omega \text{ is compactly supported}\}$$

be the set of all compactly supported smooth k-forms. Obviously

(1) if ω_1, ω_2 are compactly supported k-forms, so is $c_1\omega_1 + c_2\omega_2$;
(2) if ω is compactly supported, so is $d\omega$.

So $\Omega_c^k(M)$'s are vector spaces, and the exterior derivative makes these vector spaces a cochain complex

$$0 \to \Omega_c^0(M) \xrightarrow{d} \Omega_c^1(M) \xrightarrow{d} \Omega_c^2(M) \xrightarrow{d} \Omega_c^3(M) \xrightarrow{d} \cdots.$$

As in the ordinary de Rham theory, we denote

$$Z_c^k(M) = \{\omega \in \Omega_c^k(M) \mid d\omega = 0\},$$
$$B_c^k(M) = \{\omega \in \Omega_c^k(M) \mid \omega = d\eta \text{ for some } \eta \in \Omega_c^{k-1}(M)\}.$$

Definition 1.1. The *k-th de Rham cohomology group with compact supports* of M is

$$H_c^k(M) = \frac{Z_c^k(M)}{B_c^k(M)}.$$

We start with several remarks, mainly indicating the relation and difference between the compact supported de Rham cohomology groups and the ordinary de Rham cohomology groups.

Remark. Let M be a smooth manifold.

(1) If M is compact, then $\Omega_c^k(M) = \Omega^k(M)$ for all k, and thus

$$H_c^k(M) = H_{dR}^k(M), \qquad \forall k.$$

(2) By definition we have $Z_c^k(M) = Z^k(M) \cap \Omega_c^k(M)$. However, in general,

$$B_c^k(M) \neq B^k(M) \cap \Omega_c(M).$$

(Can you find an example?)

(3) As in the ordinary case, we can define a *cup product*

$$\cup : H_c^k(M) \times H_c^l(M) \to H_c^{k+l}(M), \quad (\omega, \eta) \mapsto [\omega \wedge \eta]$$

which makes $H_c^*(M) = \oplus_{k=1}^n H_c^k(M)$ a graded ring.

(In fact, the cup product is also well-defined as $\cup : H_c^k(M) \times H_{dR}^l(M) \to H_c^{k+l}(M)$ and as $\cup : H_{dR}^k(M) \times H_c^l(M) \to H_c^{k+l}(M)$.)

(4) For $k = 0$, by definition

$$H_c^0(M) = Z_c^0(M) = \{f \in C^\infty(M) \mid df = 0 \text{ and } \mathrm{supp}(f) \text{ is compact}\}.$$

But $df = 0$ if and only if f is locally constant, i.e. f is constant on each connected component. On the other hand, a locally constant function has to be zero on any non-compact connected component. So we conclude

$$H_c^0(M) \simeq \mathbb{R}^{m_c},$$

where m_c is the number of *compact* connected components of M.

(5) In particular, we see $H_c^0(\mathbb{R}^n) = 0$ for $n \geq 1$, and $H_c^0(pt) = \mathbb{R}$, where pt is a one point set. Since $\mathbb{R}^n$ is homotopy equivalent to $\{pt\}$, we conclude

$$\boxed{H_c^k(M)\text{'s are no longer homotopy invariants.}}$$

(6) Now let $\varphi : M \to N$ be a smooth map. Then by definition,

$$\mathrm{supp}(\varphi^*\omega) \subset \varphi^{-1}(\mathrm{supp}(\omega)).$$

So if $\omega \in \Omega_c^k(N)$, in general we may have $\varphi^*\omega \notin \Omega_c^k(M)$. In particular, we can not pull back compactly-supported cohomology classes on N to compactly-supported cohomology classes on M. In the ordinary theory, we used the pull-back to prove the homotopy invariance and to construct the M-V sequence. It turns out that in the "compactly-supported theory", we can still do two rescues, one can be used to get a partial result on homotopy invariance, while the other can be used to construct a new M-V sequence.

<u>Rescue A</u>. If $\varphi : M \to N$ is *proper*, then the pull-back $\varphi^*\omega$ of a compactly supported differential form $\omega \in \Omega_c^k(N)$ is a compactly supported differential form on M. So the map

$$\varphi^* : H_c^k(N) \to H_c^k(M)$$

is still well-defined. In this case one can prove

Theorem 1.2. *If $\varphi_0, \varphi_1 : M \to N$ are proper smooth maps that are* properly homotopic, *then the induced maps*

$$\varphi_1^* = \varphi_2^* : H_c^k(N) \to H_c^k(M).$$

Note that any diffeomorphism is proper. So in particular we get

Corollary 1.3. *If M is diffeomorphic to N, then $H_c^k(M) = H_c^k(N)$.*

Rescue B. If $\iota : U \to M$ is an *open* submanifold, then we can define a "push-forward" map

$$\iota_* : \Omega_c^k(U) \to \Omega_c^k(M)$$

which extend a compactly supported k-form on U to a compactly supported k-form on M by performing the "zero-extension". It is easy to see

$$d\iota_* = \iota_* d,$$

so ι_* descends to a linear map

$$\iota_* : H_c^k(U) \to H_c^k(M).$$

Using the push-forward, one can construct the M-V sequence for de Rham cohomology with compact supports:

Theorem 1.4. *Suppose U, V are open subsets in M so that $M = U \cup V$. Then there exists linear maps $\delta_k : H_c^k(M) \to H_c^{k+1}(U \cap V)$ so that the following sequence is exact*

$$\cdots \xrightarrow{\delta_{k-1}} H_c^k(U\cap V) \xrightarrow{\alpha_k} H_c^k(U)\oplus H_c^k(V) \xrightarrow{\beta_k} H_c^k(M) \xrightarrow{\delta_k} H_c^{k+1}(U\cap V) \xrightarrow{\alpha_{k+1}} \cdots,$$

where α_k and β_k are given by

$$\alpha_k([\omega]) = ((\jmath_1)_*[\omega], -(\jmath_2)_*[\omega])$$

and

$$\beta_k([\omega_1],[\omega_2]) = (\iota_1)_*[\omega_1] + (\iota_2)_*[\omega_2].$$

Note that on each level k, the "direction" of this new M-V sequence is opposite to that of the M-V sequences for the usual de Rham cohomology groups.

Example. Consider $M = \mathbb{R}^n$ $(n \geq 1)$. We have seen

$$H_c^0(\mathbb{R}^n) = 0.$$

Now we show that for any $1 \leq k < n$, one has

$$H_c^k(\mathbb{R}^n) = 0.$$

We identify $\mathbb{R}^n$ with $S^n - \{N\}$, where N is the north pole. Then we get an "inclusion" map $\iota : \mathbb{R}^n \to S^n$, and the words "compactly supported in $\mathbb{R}^n$" is equivalent to "supported in a subset of S^n that is away from N".

- Case $1 = k < n$. Take any $\omega \in Z_c^1(\mathbb{R}^n)$. Then $\iota_*\omega \in Z^1(S^n)$ which is supported in $S^n - U$ for some neighborhood U of p. Since $H^1(S^n) = 0$, the close 1-form $\iota_*\omega$ is exact, i.e. there exists $\eta \in \Omega^0(S^n) = C^\infty(S^n)$ so that $\omega = d\eta$. Moreover, the fact $d\eta = \iota_*\omega = 0$ on U implies that η equals some constant c on U. It follows that if we take $\tilde\eta = \eta - c$, then $\tilde\eta \in \Omega_c^0(S^n - \{N\}) = \Omega_c^0(\mathbb{R}^n)$ and $d\tilde\eta = \omega$.

- case $1 < k < n$ Again we take $\omega \in Z_c^k(\mathbb{R}^n)$ and consider $\iota_*\omega \in Z^k(S^n)$, which is supported in some $S^n - U$. Since $H_{dR}^k(S^n) = 0$, one can find $\eta \in \Omega^{k-1}(S^n)$ such that $\omega = d\eta$. By shrinking the neighborhood U of p, we can assume that U is contractible. Then the fact $d\eta = \omega = 0$ in U implies that η is exact in U, i.e. one can find a $\mu \in \Omega^{k-2}(U)$ such that $\eta = d\mu$. Now one pick a bump function ρ on S^n which vanishes on $S^n - U$ and equals 1 near p. Then $\tilde{\eta} = \eta - d(\rho\mu) \in \Omega^{k-1}(S^n)$ and $\tilde{\eta} = 0$ near p, i.e. it defines a compactly supported $(k-1)$-form on $\mathbb{R}^n$. By construction, $d\tilde{\eta} = d\eta = \omega$.

  Question: What if $k = n$? Does the same arguments work?

2. The de Rham cohomology groups of top degree

We have computed $H_c^k(\mathbb{R}^n)$ for $k < n$. Now we compute $H_c^n(\mathbb{R}^n)$. The key ingredient is: we can integrate any compactly-supported smooth n-form. We start by the $n = 1$ case:

Example. Let's try to compute $H_c^1(\mathbb{R})$. To do so we consider the integration map

$$\int_{\mathbb{R}} : Z_c^1(\mathbb{R}) = \Omega_c^1(\mathbb{R}) \to \mathbb{R}, \quad \omega \mapsto \int_{\mathbb{R}} \omega.$$

This map is clearly linear and surjective. Moreover, it vanishes on $B_c^1(\mathbb{R})$ (by the fundamental theorem of calculus), so it induces a surjective linear map

$$\int_{\mathbb{R}} : H_c^1(\mathbb{R}) \to \mathbb{R}.$$

Moreover, if $\int_{\mathbb{R}} f(t)dt = 0$, where $f \in C_c^\infty(\mathbb{R})$, then the function $g(t) = \int_{-\infty}^t f(\tau)d\tau$ is smooth and compactly supported and $dg = f(t)dt$. In other words, $f(t)dt \in B_c^1(\mathbb{R})$, i.e. $[f(t)dt] = 0$. It follows that $\int_{\mathbb{R}}$ is an isomorphism between $H_c^1(\mathbb{R})$ and $\mathbb{R}$. So

$$H_c^1(\mathbb{R}) \simeq \mathbb{R}.$$

Essentially the same method works in higher dimension. Of course, since we want to use integral of top forms, we need to assume that the manifold is orientable. It is not surprising that we will need to use Stokes' formula.

Now let M be an n-dimensional connected oriented (= orientable with an orientation chosen) manifold. Consider the map

$$\int_M : \Omega_c^n(M) \to \mathbb{R}, \quad \omega \mapsto \int_M \omega.$$

Now suppose $\omega \in B_c^k(M)$, i.e. $\omega = d\eta$ for some $\eta \in \Omega_c^{n-1}(M)$. We can take a compact set K in M with smooth boundary, so that $K \supset \mathrm{supp}(\eta)$. Then by the Stokes' formula,

$$\int_M \omega = \int_M d\eta = \int_K d\eta = \int_{\partial K} \eta = 0.$$

So $\int_M$ induces a linear map

$$\int_M : H_c^n(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega.$$

Proposition 2.1. *Suppose M is oriented. Then the map $\int_M : H_c^n(M) \to \mathbb{R}$ described above is surjective.*

Proof. Fix a volume form ω on M. For any c, one can find a smooth function f that is compactly supported in a coordinate chart U, such that $\int f\omega = c$. $\square$

The surjectivity of this map has an important consequence:

Corollary 2.2. *If $\omega \in \Omega^n(S^n)$ and $\int_{S^n} \omega = 0$, then ω is exact.*

Proof. We have seen that the map $\int_M$ is linear and surjective. But

$$H_c^n(S^n) = H^n(S^n) \simeq \mathbb{R}.$$

So $\int_M$ must be a linear isomorphism. In other words, if $\int_{S^n} \omega = 0$, then $[\omega] = 0$, i.e. ω is exact. $\square$

Using this, we can prove the following Poincaré lemma for compactly supported de Rham cohomology:

Theorem 2.3 (Poincaré lemma)**.** $H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n, \\ 0, & k \neq n. \end{cases}$

Proof. It remains to prove: for $n = k \geq 2$, the surjective linear map

$$\int_{\mathbb{R}^n} : H_c^n(\mathbb{R}^n) \to \mathbb{R}.$$

is an isomorphism. More precisely, we need to prove that if $\int_{\mathbb{R}^n} \omega = 0$ for some $\omega \in \Omega_c^n(\mathbb{R}^n) = Z_c^n(\mathbb{R}^n)$, then $\omega \in B_c^n(\mathbb{R}^n)$. As before we consider the "inclusion" map $\iota : \mathbb{R}^n \to S^n$. Then for any $\omega \in \Omega_c^n(\mathbb{R}^n)$, we have $\iota_*\omega \in \Omega^n(S^n)$. Since

$$\int_{S^n} \iota_*\omega = \int_{\mathbb{R}^n} \omega = 0,$$

By Corollary 2.2 we see $\iota_*\omega = d\eta$ for some $\eta \in \Omega^{n-1}(S^n)$.

The rest of proof is identically the same as before: We take an open contractible neighborhood U of p in S^n on which ω vanishes. Then we adjust η to $\tilde{\eta} = \eta - d(\rho\mu)$, where ρ is a bump function and $\mu \in \Omega^{k-2}(U)$ is such that $d\mu = \eta$ on U. Then $\tilde{\eta}$ is a compactly supported $(n-1)$-form on $\mathbb{R}^n$ and $d\tilde{\eta} = \omega$. $\square$

As an immediate corollary, we get

Corollary 2.4. *If M admits a finite good cover, then* $\dim H_c^k(M) < \infty$ *for all k.*

Proof. Use Mayer-Vertoris sequence and induction and the number of open sets in a good cover. The same as the proof for the ordinary de Rham cohomology that we did last time. $\square$

More generally, we have

Theorem 2.5. *For any n-dimensional connected orientable manifold M, the map $\int_M : H^n_c(M) \to \mathbb{R}$ is an isomorphism. In particular, $H^n_c(M) \simeq \mathbb{R}$.*

Proof. Since $\int_M$ is linear and surjective, it remains to prove $\int_M$ is injective, i.e.

$$\omega \in \Omega^n_c(M) \text{ such that } \int_M \omega = 0 \Longrightarrow \omega = d\mu \text{ for some } \mu \in \Omega^{n-1}_c(M). \tag{1}$$

To prove this, we can use induction on the number of open sets that is needed to cover the support of ω by a good cover.

More precisely, since M is connected and $\text{supp}(\omega)$ is compact, we can take a connected compact set $K_\omega \supset \text{supp}(\omega)$. If we can cover K_ω by a good cover which contains only one chart, then the Poincaré lemma implies (1). Now suppose (1) holds for any $\omega \in \Omega^n_c(M)$ such that K_ω can be covered by $k-1$ "good charts", and suppose $\omega \in \Omega^n_c(M)$ satisfies the property that K_ω admits a good cover $\{U_1, \cdots, U_k\}$. There exists one U_i, say U_k for simplicity, such that both $U = U_1 \cup \cdots \cup U_{k-1}$ and $V = U_k$ are connected. (Reason: One can form a graph G of k vertices $v_1, \cdots, v_k$, corresponding to $U_1, \cdots, U_k$ respectively, so that v_i is connected to v_j if and only if $U_i \cap U_j \neq \emptyset$. Then this graph G is a connected graph. By graph theory, one can always delete one vertex of a connected graph so that the remaining graph is still connected: one only need choose a spanning tree of the graph, and delete a "leaf" of the tree.) We pick a partition of unity $\{\rho_U, \rho_V\}$ of $U \cup V$ subordinate to the cover $\{U, V\}$, and let

$$\omega_U = \rho_U \omega, \qquad \omega_V = \rho_V \omega.$$

Since K_ω is connected, $U \cap V \neq \emptyset$. We pick an n-form ω_0 supported in $U \cap V$ so that

$$\int_M \omega_0 = \int_M \omega_U.$$

Then $\omega_U - \omega_0$ is supported in U which is connected and admits a good cover of $k-1$ good charts, and $\int_M (\omega_U - \omega_0) = 0$. So by the induction hypothesis,

$$\omega_U - \omega_0 = d\eta_U$$

for some $\eta_U \in \Omega^{k-1}_c(M)$. Similarly $\int_M (\omega_V + \omega_0) = -\int_M \omega_U + \int_M \omega_0 = 0$ implies

$$\omega_V + \omega_0 = d\eta_V$$

for some $\eta_V \in \Omega^{k-1}_c(M)$. It follows that

$$\omega = \omega_U + \omega_V = d(\eta_U + \eta_V),$$

where $\eta_U + \eta_V \in \Omega^{k-1}_c(M)$. □

As a corollary, we get

Corollary 2.6. *Let M be compact, connected, orientable and* $\dim M = n$. *Then*

$$H^n_{dR}(M) \simeq \mathbb{R}.$$

For orientable non-compact manifolds, one has

Theorem 2.7. *For any non-compact connected orientable manifold M of dimension n,*

$$H^n_{dR}(M) = 0$$

Proof. From PSet 1 Part 2 we have seen that for any smooth manifold M, there exists an positive smooth exhaustion function f. In other words, f is a smooth function on M so that for any c, the sub-level set $f^{-1}((-\infty, c])$ is compact in M. By adding a constant, we may assume $\inf f = 0$. Since M is connected and non-compact, we must have $f(M) = [0, \infty)$. Now for each integer k we let $V_k = f^{-1}((k-2, k))$. Then $\{V_k\}$ is a locally finite open covering of M. Let $\{\rho_k\}$ be a partition of unity subordinate to this covering. For each k, we choose some $\eta_k \in \Omega^n_c(V_k \cap V_{k+1}) \subset \Omega^n_c(M)$ so that $\int_M \eta_k = 1$.

For any $\omega \in \Omega^n(M)$, we let $\omega_k = \rho_k \omega \in \Omega^n_c(V_k)$. Let $c_1 = \int_{V_1} \omega_1$. Then $\omega_1 - c_1\eta_1 \in \Omega^n_c(V_1)$ and

$$\int_{V_1} (\omega_1 - c_1\eta_1) = 0.$$

So by Theorem 2.5, one can find $\mu_1 \in \Omega^{n-1}_c(V_1)$ so that

$$d\mu_1 = \omega_1 - c_1\eta_1.$$

Next we choose $c_2 \in \mathbb{R}$ so that $\int_{V_2}(\omega_2 + c_1\eta_1 - c_2\eta_2) = 0$, and conclude that there exists $\mu_2 \in \Omega^{n-1}_c(V_2)$ such that

$$d\mu_2 = \omega_2 + c_1\eta_1 - c_2\eta_2.$$

Continuing this process, we can find a sequence $c_k \in \mathbb{R}$ and $\mu_k \in \omega^{n-1}_c(V_k)$ so that

$$d\mu_k = \omega_k + c_{k-1}\eta_{k-1} - c_k\eta_k.$$

Let $\mu = \sum \mu_k$. Note that this is a locally finite sum, and thus defines an element in $\Omega^{n-1}(M)$. Moreover, by construction we have

$$d\mu = d\sum \mu_k = \sum \omega_k = \omega.$$

□

Remark. If M is a non-orientable connected smooth manifold of dimension n, then

$$H^n_{dR}(M) = 0 \quad \text{and} \quad H^n_c(M) = 0.$$

For a proof, c.f. Lee, page 456-457.

In summary: suppose M is a connected smooth manifold of dimension n. Then

$$H^n_{dR}(M) \simeq \begin{cases} \mathbb{R}, & M \text{ is compact and orientable,} \\ 0, & M \text{ is non-compact or non-orientable} \end{cases}$$

while

$$H^n_c(M) \simeq \begin{cases} \mathbb{R}, & M \text{ is orientable,} \\ 0, & M \text{ is non-orientable.} \end{cases}$$