LECTURE 28: VECTOR BUNDLES AND FIBER BUNDLES

1. Vector Bundles

In general, smooth manifolds are very "non-linear". However, there exist many smooth manifolds which admit very nice "partial linear structures". For example, given any smooth manifold M of dimension n, the tangent bundle

$$TM = \{(p, X_p) \mid p \in M, X_p \in T_pM\}$$

is "linear in tangent variables". We have seen in PSet 2 Problem 9 that TM is a smooth manifold of dimension $2n$ so that the canonical projection $\pi : TM \to M$ is a smooth submersion. A local chart of TM is given by

$$T\varphi = (\pi, d\varphi) : \pi^{-1}(U) \to U \times \mathbb{R}^n,$$

where $\{\varphi, U, V\}$ is a local chart of M. Note that the local chart map $T\varphi$ "preserves" the linear structure nicely: it maps the vector space $\pi^{-1}(p) = T_pM$ isomorphically to the vector space $\{p\} \times \mathbb{R}^n$. As a result, if you choose another chart $(\tilde{\varphi}, \widetilde{U}, \widetilde{V})$ containing p, then the map $T\widetilde{\varphi} \circ T\varphi^{-1} : \{p\} \times \mathbb{R}^n \to \{p\} \times \mathbb{R}^n$ is a linear isomorphism which depends smoothly on p.

In general, we define

Definition 1.1. Let E, M be smooth manifolds, and $\pi : E \to M$ a surjective smooth map. We say (π, E, M) is a *vector bundle of rank r* if for every $p \in M$, there exits an open neighborhood U_α of p and a diffeomorphism (called the *local trivialization*)

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

so that

(1) $E_p = \pi^{-1}(p)$ is a r dimensional vector space, and $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^r$ is a linear map.
(2) For $U_\alpha \cap U_\beta \neq \emptyset$, there is a smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ so that

$$\Phi_\beta \circ \Phi_\alpha^{-1}(m, v) = (m, g_{\beta\alpha}(m)(v)), \quad \forall m \in U_\alpha \cap U_\beta, v \in \mathbb{R}^r.$$

We will call E the *total space*, M the *base* and $\pi^{-1}(p)$ the *fiber* over p. (In the case there is no ambiguity about the base, we will denote a vector bundle by E for short.)

(Roughly speaking, a vector bundle E over M is "a smooth varying family of vector spaces parameterized by a base manifold M".)

Remark. A vector bundle of rank 1 is usually called a *line bundle*.

Remark. It is easy to define conception of a *vector sub-bundle*: a vector bundle (π_1, E_1, M) is a sub-bundle of (π, E, M) if $(E_1)_p \subset E_p$ for any p, and $\pi_1 = \pi|_{E_1}$.

Example. Here are some known examples:

(1) For any smooth manifold M, $E = M \times \mathbb{R}^r$ is a *trivial bundle* over M.
(2) The tangent bundle TM and the cotangent bundle T^*M are both vector bundles over M.
(3) Given any smooth submanifold $X \subset M$, the normal bundle
$$NX = \{(p, v) \mid p \in X, v \in N_pX\},$$
(where N_pX is the quotient vector space T_pM/T_pX) is a vector bundle over X. Note: NX is NOT a vector sub-bundle of TM.
(4) Any rank r distribution $\mathcal{V}$ on M is a rank r vector bundle over M. It is a vector *sub-bundle* of the tangent bundle TM.

Example. The *canonical line bundle* over $\mathbb{RP}^n = \{l \text{ is a line through } 0 \text{ in } \mathbb{R}^{n+1}\}$ is
$$\gamma_n^1 = \{(l, x) \mid l \in \mathbb{RP}^n, x \in l \subset \mathbb{R}^{n+1}\}.$$
(Can you write down a local trivialization?)

In particular if $n = 1$, we have $\mathbb{RP}^1 \simeq S^1$. In this case the canonical line bundle γ_1^1 is nothing else but the infinite Möbius band, which is a line bundle over S^1.

Another way to obtain the infinite Möbius band γ_1^1 is to identify S^1 with $[0, 1]$, with end points 0 and 1 "glued together". Then the Möbius band is $[0, 1] \times \mathbb{R}$, with "boundary lines $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$ glued together" via $(0, t) \sim (1, -t)$.

Example. One can extend operations on vector spaces to operations on vector bundles.

(1) Given any vector bundle (π, E, M), one can define the dual vector bundle by replacing each E_p with its dual E_p^*. (How to define local trivializations? What are the transition maps $g_{\beta\alpha}$'s?) For example, T^*M is the dual bundle of TM.
(2) Let (π_1, E_1, M) and (π_2, E_2, M) be two vector bundles over M of rank r_1 and r_2 respectively. Then the *direct sum bundle* $(\pi_1 \oplus \pi_2, E_1 \oplus E_2, M)$ is the rank $r_1 + r_2$ vector bundle over M with fiber $(\pi_1 \oplus \pi_2)^{-1}(p) = (E_1)_p \oplus (E_2)_p$. (How to define local trivializations? What are the transition maps $g_{\beta\alpha}$'s?)
(3) Then the *tensor product bundle* $(\pi_1 \otimes \pi_2, E_1 \otimes E_2, M)$ is the rank r_1r_2 vector bundle over M with fiber $(\pi_1 \otimes \pi_2)^{-1}(p) = (E_1)_p \otimes (E_2)_p$. (How to define local trivializations? What are the transition maps $g_{\beta\alpha}$'s?) For example, we have the (k, l)-*tensor bundle* $\otimes^{k,l}TM := (TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$ over M.
(4) Similarly one can define the *exterior power bundle* Λ^kT^*M, whose fiber at point $p \in M$ is the linear space $\Lambda^kT_p^*M$. (Local trivializations? Transition maps?)
(5) Let $f : N \to M$ be a smooth map, and (π, E, M) a vector bundle over M. Then one can define a *pull-back bundle* f^*E over N by setting the fiber over $x \in N$ to be the fiber of $E_{f(x)}$. (Local trivializations? Transition maps?)

In particular, the restriction of a vector bundle (π, E, M) to a submanifold N of the base manifold M is a vector bundle over the submanifold N. (Note: this is not a vector sub-bundle of (π, E, M)!)

We have seen that any smooth manifold can be embedded into the simplest manifold: an Euclidian space. A natural question is whether the same conclusion holds for vector bundles? More precisely, can we embed any vector bundle (π, E, M) into some trivial bundle $M \times \mathbb{R}^N$ as a vector sub-bundle? The answer is yes if M is compact, and the proof is similar to the proof of the "simple Whitney embedding theorem" in Lecture 9. (The conclusion could be wrong if M is non-compact.)

Theorem 1.2. *If M is compact, then any vector bundle E over M is isomorphic to a sub-bundle of a trivial vector bundle over M.*

Proof. (Please compare this proof with the proof of Theorem 1.1 in Lecture 9). Take a finite open cover $\{U_i\}_{1\le i\le k}$ of M so that E is trivial over each U_i via the trivialization maps $\Phi_i = (\pi, \Phi_i^2) : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^r$. Let $\{\rho_i\}_{1\le i\le k}$ be a P.O.U. subordinate to $\{U_i\}_{1\le i\le k}$. Consider

$$\Phi : E \to M \times (\mathbb{R}^r)^k, \quad v \mapsto (\pi(v), \rho_1(\pi(v))\Phi_1^2(v), \cdots, \rho_k(\pi(v))\Phi_k^2(v)).$$

Note that on each fiber E_p, Φ is a linear isomorphism onto its image. The conclusion follows, since the image of Φ is a vector sub-bundle of $M \times \mathbb{R}^{rk}$: for any $p \in M$, there exists i so that $\rho_i(p) \neq 0$. Then the map Φ_i^2 induces a local trivialization near p. □

As a corollary we get the following fundamental fact in topological K-theory:

Corollary 1.3. *If M is compact, then for any vector bundle E over M, there exists a vector bundle F over M so that $E \oplus F$ is a trivial bundle over M.*

Proof. We have seen that E is a vector sub-bundle of a trivial bundle $M \times \mathbb{R}^N$ over M. Now we put an inner product on $\mathbb{R}^N$, and take the fiber F_p of F at $p \in M$ to be the orthogonal complement of E_p in $\mathbb{R}^N$. (One should check that F is a vector bundle over M.) □

2. Sections of vector bundles

The following definition is natural:

Definition 2.1. A *(smooth) section* of a vector bundle (π, E, M) is a (smooth) map $s : M \to E$ so that $\pi \circ s = \mathrm{Id}_M$. The set of all sections of E is denoted by $\Gamma(E)$, and the set of all smooth sections of E is denoted by $\Gamma^\infty(E)$.

Remark. Obviously if s_1, s_2 are smooth sections of E, so is $as_1 + bs_2$. So $\Gamma^\infty(E)$ is an (infinitely dimensional) vector space. In fact, one can say more: if s is a smooth section of E and f is a smooth function on M, then fs is a smooth section of E. So $\Gamma^\infty(E)$ is a $C^\infty(M)$-module. (According to the famous Serre-Swan theorem, there is an equivalence of categories between that of finite rank vector bundles over M and finitely generated projective modules over $C^\infty(M)$.)

Remark. Many geometrically interesting objects on M are defined as smooth sections of some (vector) bundles over M. For example,

- A smooth *vector field* on M = a smooth section of TM.
- A smooth (k,l)-*tensor field* on M = a smooth section of $\otimes^{k,l}TM$.
- A smooth k-*form* on M = a smooth section of $\Lambda^k T^*M$.
- A *volume form* on M = a non-vanishing smooth section of $\Lambda^n T^*M$.
- A *Riemannian metric* on M = a smooth section of $\otimes^2 T^*M$ satisfying some extra (symmetric, positive definite) conditions
- A *symplectic form* on M = a smooth section of $\Lambda^2 T^*M$ satisfying some extra (closed, non-degenerate) conditions

By definition any vector bundle admits a trivial smooth section: the zero section

$$s_0 : M \to E, \quad p \mapsto (p, 0).$$

On the other hand, it is possible that a vector bundle admits no non-vanishing section. For example, we have seen

- M is orientable if and only if $\Lambda^n T^*M$ admits a global non-vanishing section.
- TS^{2n} admits no non-vanishing section.

Here is another example:

Example. Consider the canonical line bundle γ_n^1 over $\mathbb{RP}^n$. We claim that there is no non-vanishing smooth section $s : \mathbb{RP}^n \to \gamma_n^1$. To see this we consider the composition

$$\varphi : S^n \xrightarrow{P} \mathbb{RP}^n \xrightarrow{s} \gamma_n^1,$$

where P is the projection $p(\pm x) = l_x$, the line through x. By definition, φ is of form

$$x \mapsto (l_x, f(x)x),$$

where f is a smooth function on S^n satisfying $f(-x) = -f(x)$. By mean value property, f vanishes at some x_0. It follows that s vanishes at x_0 also.

Although there might be no non-vanishing global sections, locally there are plenty of non-vanishing sections. Let E be a rank r vector bundle over M, and U an open set in M.

Definition 2.2. A *local frame* of E over U is an ordered r-tuple $s_1, \cdots, s_r$ of smooth section of E over U so that for each $p \in U$, $s_1(p), \cdots, s_r(p)$ form a basis of E_p.

Example. Let M be a smooth manifold and U be a coordinate patch. Then

- $\partial_1, \cdots, \partial_n$ form a local frame of TM over U.
- $dx^1, \cdots, dx^n$ form a local frame of T^*M over U.

The following fact is basic. We will leave the proof as an exercise.

Proposition 2.3. *Let E be a smooth vector bundle over M.*

(1) *A section $s \in \Gamma(E)$ is smooth if and only if for any $p \in M$, there is a neighborhood U of p and a local frame $s_1, \cdots, s_r$ of E over U so that $s = c_1 s_1 + \cdots + c_r s_r$ for some smooth functions $c_1, \cdots, c_r$ defined in U.*
(2) *E is a trivial bundle if and only if there exists a global frame of E on M.*

3. De Rham cohomology groups of vector bundles

We have seen

$$H^k_{dR}(M \times \mathbb{R}^r) \simeq H^k_{dR}(M) \qquad \text{and} \qquad H^k_c(M \times \mathbb{R}^r) \simeq H^{k-r}_c(M).$$

A vector bundle E can be viewed as a "twisted product" of a smooth manifold M (the base) with a vector space (the fiber). So it is natural to study the relation between the cohomology groups of E and the cohomology groups of M.

Proposition 3.1. *For any vector bundle E over M, one has*

$$H^k_{dR}(E) = H^k_{dR}(M), \qquad \forall k.$$

Proof. This is a consequence of the homotopy invariance: E is homotopy equivalent to M, since if we let $s_0 : M \to E$ be the zero section, then $\pi \circ s_0 = \mathrm{Id}_M$, and $s_0 \circ \pi \sim \mathrm{Id}_E$ via the homotopy

$$F : E \times \mathbb{R} \to E, \qquad (x, v, t) \mapsto (x, tv).$$

□

In general, the same result fails for compact supported cohomology groups. For example, let E be the infinite Möbius band, which is a line bundle over S^1. Since E is non-orientable, one has

$$H^2_c(E) = 0 \not\simeq \mathbb{R} \simeq H^1_{dR}(S^1) = H^1_c(S^1).$$

On the other hand, if we assume orientablity, then

Proposition 3.2. *Let E be a rank r vector bundle over M. Assume both E and M are oriented with finite dimensional compact supported de Rham cohomology groups, then*

$$H^k_c(E) \simeq H^{k-r}_c(M), \qquad \forall k.$$

Proof. We apply Poincaré duality twice:

$$H^k_c(E) \simeq H^{n+r-k}_{dR}(E) \simeq H^{n+r-k}_{dR}(M) \simeq H^{k-r}_c(M).$$

□

Now let M be an oriented, compact and connected smooth n-manifold, and E an oriented vector bundle of rank r over M. The constant function 1 on M gives us a degree-0 cohomology class $[1] \in H^0_{dR}(M)$. So the isomorphism above produces an element

$$[T] \in H^r_c(E).$$

Definition 3.3. We call $[T]$ the *Thom class* of (π, E, M).

Remark. The isomorphism $H^k_c(E) \simeq H^{k-r}_c(M)$ is called the *Thom duality*. It is given explicitly by an *integration along the fibers* map

$$\pi_* : \Omega^k_c(E) \to \Omega^{k-r}(M).$$

More precisely, on a local trivialization we can pick coordinates $x^1, \cdots, x^n$ of M and $s^1, \cdots, s^r$ of the fiber. For any k-form ($k \geq r$) ω on M, locally we can write

$$\omega = f(x, s)(\pi^*\theta) \wedge ds^1 \cdots \wedge ds^r + \text{terms with less fiber 1-forms},$$

where θ is a $(k-r)$-form on M, and f is compactly supported. Then

$$\pi_*\omega := \theta \int_{\mathbb{R}^r} f(x, s)\ ds^1 \cdots ds^r.$$

One can check that

- $\pi_*\omega$ is well-defined.
  - To prove the independence of the choices of the fiber variables $s^1, \cdots, s^r$, one need the following fact: Since E and M are both orientable, one can always choose fiber variables so that the transition maps $g_{\alpha\beta} \in \mathrm{GL}^+(n, \mathbb{R})$.
- $d\pi_* = \pi_* d$. ($\Longrightarrow \pi_*$ induces a linear map $\pi_* : H_c^k(E) \to H_c^{k-c}(M)$.)
- The induced map $\pi_* : H_c^k(E) \to H_c^{k-c}(M)$ is an isomorphism.
- Moreover, one has the projection formula

$$\pi_*(\pi^*\theta \wedge \omega) = \theta \wedge \pi_*\omega, \qquad \forall \theta \in \Omega^*(M) \text{ and } \omega \in \Omega_c^*(E).$$

The Thom class gives us the "inverse" to the "integration along fibers map". More precisely, by definition, $\pi_*([T]) = 1$. By using the projection formula above, one has

$$(\pi_*)^{-1}([\omega]) = [\pi^*\omega] \cup [T], \qquad \forall \omega \in Z_c^*(E).$$

Let E be an oriented vector bundle of rank r over an oriented connected compact smooth manifold M. Let $s : M \to E$ be any global section of E. Then by using the pull-back, we get an element

$$s^*([T]) \in H_{dR}^r(M).$$

Proposition 3.4. *The de Rham cohomology class $s^*([T])$ is independent of the choices of s.*

Proof. Let $s_0 : M \to E$ be the zero section. Then $s_0 \sim s$ via the homotopy

$$F : M \times \mathbb{R}, (m, t) \mapsto (m, ts(m)).$$

So $s^*([T]) = s_0^*([T])$. □

Definition 3.5. The de Rham cohomology class

$$\chi(E) := s^*([T]) \in H_{dR}^r(M)$$

is called the *Euler class* of E.

The Euler class of a vector bundle is an obstruction to the existence of a non-vanishing global section.

Theorem 3.6. *If an oriented vector bundle E over an oriented compact connected smooth manifold M admits a non-vanishing global section, then $\chi(E) = 0$.*

Proof. Let $s : M \to E$ be an non-vanishing section. Take $T \in Z_c^r(E)$ so that $[T]$ is the Thom class. Take $c \in \mathbb{R}$ so that the section $s_1 = cs$ does not intersect $\mathrm{supp}(T)$. It follows

$$\chi(E) = s_1^*([T]) = [s_1^* T] = 0.$$

□

Finally we remark that for $E = TM$ the tangent bundle, one has

$$\chi(TM) = \chi(M)[\mu],$$

where $\mu \in \Omega^n(M)$ is any top form with $\int_M \mu = 1$, and

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H_{dR}^i(M)$$

is the Euler characteristic number of M. This explains the name "Euler class". Moreover, this explains why any smooth vector field on S^{2n} admits at least one zero, while there exists non-vanishing smooth vector fields on S^{2n+1}:

$$\chi(S^{2n}) \neq 0 \qquad \text{while} \qquad \chi(S^{2n+1}) = 0.$$