LECTURE 29: CONNECTIONS AND CURVATURES

1. Connections on vector bundles

Let E be a vector bundle over M, and s a smooth section. We would like to "differentiate" s. Let's first consider the case that $E = M \times \mathbb{R}^r$. In this case, a section s is just a vector-valued function

$$s = (f_1, \cdots, f_r)^T,$$

where $f_i \in C^\infty(M)$ are smooth functions. It is naturally to "define" ds to be

$$ds = (df_1, \cdots, df_r)^T.$$

This is a "vector-valued 1-form", in the sense that if we plug-in a smooth vector field X on M into ds, we will get a vector-valued function

$$ds(X) = (df_1(X), \cdots, df_r(X))^T \in C^\infty(M, \mathbb{R}^r).$$

So to "differentiate" smooth sections of vector bundles, we are naturally led to extend the conception of $\mathbb{R}$-valued differential forms to vector-valued differential forms.

Recall that a k-tensor T on V is a multi-linear map

$$T : V \times V \times \cdots \times V \to \mathbb{R}.$$

For any vector space W, one may also study W-valued k-tensor, i.e. multi-linear map

$$T : V \times V \times \cdots \times V \to W.$$

Similarly one may define W-valued linear k-form, which are anti-symmetric W-valued k-tensor. Obviously they are not new objects at all: if we choose a basis of W, then we can write down T as a vector of $\mathbb{R}$-valued k-tensors or $\mathbb{R}$-valued linear k-forms.

In particular, if M is a smooth manifold, then we can talk about W-valued k-forms, which are pointwise multi-linear anti-symmetric maps

$$\eta_p : T_pM \times T_pM \times \cdots \times T_pM \to W$$

which depends on p smoothly. In other words, it is a bundle morphism (=fiberwise linear)

$$\eta : TM \otimes TM \otimes \cdots \otimes TM \to M \times W.$$

which is anti-symmetric (c.f. PSet 6 Problem 2(c)). If we allow the target vector space W to vary as the base point p changes, i.e. we want W to be E_p at p, where E is a vector bundle over M, we will get *E-valued differential forms* (where E is a vector bundle over M), which are anti-symmetric bundle morphisms

$$\eta : TM \otimes TM \otimes \cdots \otimes TM \to E.$$

Recall that any k-form on M can also be described as a smooth section of the exterior power bundle $\Lambda^k T^*M$. Similarly an E-valued k-form on M can be described as a section of the tensor product bundle $\Lambda^k T^*M \otimes E$. We take this as our definition:

Definition 1.1. Let E be any smooth vector bundle over M. We call any smooth section of $\Lambda^k T^*M \otimes E$ an *E-valued k-form* on M. The set of all E-valued k-forms is denoted by $\Omega^k(M;E)$.

Of course locally on a small open set U in M, any element $\eta \in \Omega^k(U; E|_U)$ can be written as a linear combination of elements of the form $\omega \otimes s$, where $\omega \in \Omega^k(U)$ and $s \in \Gamma^\infty(E|_U)$.

Remark. Note that in general one can no longer define the wedge product between two E-valued differential forms: what can $(\omega_1 \otimes s_1) \wedge (\omega_2 \otimes s_2)$ be? We know $\omega_1 \wedge \omega_2$ gives us a differential form on M, but we don't know how to put sections s_1 and s_2 of E together *algebraically* to get a new section of E. However, we have two rescue:

- One can define a wedge product $\wedge : \Omega^k(M) \times \Omega^l(M;E) \to \Omega^{k+l}(M;E)$ by extending the following rule linearly:
$$\omega_1 \wedge (\omega_2 \otimes s) := (\omega_1 \wedge \omega_2) \otimes s.$$
Similarly one can define the wedge product $\wedge : \Omega^l(M;E) \otimes \Omega^k(M) \to \Omega^{k+l}(M;E)$. (Thus $\Omega^*(M;E)$ is a graded (left- and right-)module over the graded algebra $\Omega^*(M)$.)
- If the fiber E_p's are not just vector spaces, but in fact algebras (so that one can "multiply" vectors in each E_p), then one can define the wedge products between elements in $\Omega^k(M;E)$ and $\Omega^l(M;E)$. This is the case, for example, for $\Omega^k(M;\mathrm{End}(E))$:
$$\wedge : \Omega^k(M;\mathrm{End}(E)) \times \Omega^l(M;\mathrm{End}(E)) \to \Omega^{k+l}(M;\mathrm{End}(E)),$$
$$(\omega_1 \otimes s_1) \wedge (\omega_2 \otimes s_2) := (\omega_1 \wedge \omega_2) \otimes (s_1 \circ s_2).$$
(Recall: $\mathrm{End}(E)$ is the vector bundle over M whose fiber at p is $\mathrm{End}(E_p)$. Since $\mathrm{End}(E_p) \simeq E_p \otimes E_p^*$ (c.f. PSet 6 Problem 2(b)), we have $\mathrm{End}(E) \simeq E \otimes E^*$.) Note that if E has rank r, then each $\mathrm{End}(E_p)$ can be identified with the general linear algebra $\mathfrak{gl}(r,\mathbb{R})$.

Our next goal is to extend the conception of exterior derivative to E-valued differential forms. We start with E-valued 0-forms, i.e. sections of E.

As we have seen, if $E = M \times \mathbb{R}^r$, then we can differentiate a section $s = (f_1, \cdots, f_r)^T$ via the formula
$$ds := (df_1, \cdots, df_r)^T,$$
which, in our new terminology, is a E-valued 1-form. This seems to be the most natural definition. However, if E is a trivial rank r-bundle over M, but not already in the product form $M \times \mathbb{R}^r$, then we may have many different ways to trivialize E: For example, if $s_1, \cdots, s_r$ is a global trivialization of E, and if $g_1, \cdots, g_r$ are positive functions on M, then $s_1/g_1, \cdots, s_r/g_r$ is another trivialization of E. Using the trivialization $s_1, \cdots, s_r$, one may identify a section s as a vector $(f_1, \cdots, f_r)^T$ if $s = \sum f_i s_i$. But if we use the trivialization $s_1/g_1, \cdots, s_r/g_r$, then we has to identify the same section s with the vector $(g_1 f_1, \cdots, g_r f_r)$.

As a result, the differential ds can be defined to be $(df_1, \cdots, df_r)^T$ in the frame $s_1, \cdots, s_r$, which is

$$\text{“}ds = \text{''} df_1 \otimes s_1 + \cdots + df_r \otimes s_r,$$

or ds can be defined to be $(d(g_1 f_1), \cdots, d(g_r f_r))^T$ in the frame $s_1/g_1, \cdots, s_r/g_r$, which is

$$\begin{aligned}\text{“}ds = \text{''} & d(g_1 f_1) \otimes s_1/g_1 + \cdots + d(g_r f_r) \otimes s_r/g_r \\ = & df_1 \otimes s_1 + (dg_1) f_1/g_1 \otimes s_1 + \cdots + df_r \otimes s_r + (dg_r) f_r/g_r \otimes s_r.\end{aligned}$$

So we conclude that even for trivial bundles, we can define different ds's, which are different but are "natural" with respect to different choices of global trivialization.

Two implications that we can see from the above example:

(1) There is no "god-given" way to differentiate a section. We could have many different ways. It is given by an extra structure on E (called a linear connection below). To differentiate sections, we have to choose a linear connection first.
(2) For a general vector bundle, there is no global frame in general. However, we can always choose local frames and define ds locally. Then we are naturally led to study "the same linear connection in different frame".

So if E is an arbitrary vector bundle, and s a section of E, then after applying a linear connection ∇, we should get an E-valued 1-form ∇s. Moreover, using either "definition" of ds above, we can check that the Leibnitz rule still holds:

$$\text{“}d(fs)\text{''} = df \otimes s + f ds.$$

So we define

Definition 1.2. Let E be a vector bundle over M. A *linear connection* on E is a linear map

$$\nabla : \Omega^0(M; E) = \Gamma^\infty(E) \to \Gamma^\infty(T^*M \otimes E) = \Omega^1(M; E)$$

such that for any $f \in C^\infty(M)$ and any $s \in \Gamma^\infty(E)$, we have

$$\nabla(fs) = df \otimes s + f \nabla s.$$

Example. If $E = M \times \mathbb{R}^r$ is a trivial vector bundle, then any section in $\Gamma^\infty(E)$ is of the form $s = (f_1, \cdots, f_r)^T$, where each f_i is a smooth function on M. In this case one can define a trivial connection ∇^0 by

$$\nabla^0(f_1, \cdots, f_r) := (df_1, \cdots, df_r)^T \in \Gamma^\infty(T^*M \otimes E).$$

Remark. If ∇^0 and ∇^1 are linear connections on E, then for any $\rho \in C^\infty(M)$,

$$\nabla := \rho \nabla^0 + (1 - \rho) \nabla^1$$

is again a linear connection on E. As a consequence, one can easily prove the existence of linear connections by using "trivial connections on trivialization neighborhoods" and partition of unity. (WARNING: If ∇_1, ∇_2 are linear connections, $\nabla_1 + \nabla_2$ and $\lambda \nabla_1$ are no longer linear connections in general.)

Remark. A connection is not a tensor since it is not $C^\infty(M)$-linear. However, if ∇^0 and ∇^1 are two linear connections on E, then one can check that

$$A := \nabla^1 - \nabla^0 : \Gamma^\infty(E) \to \Gamma^\infty(T^*M \otimes E)$$

satisfies $A(fs) = fA(s)$:

$$A(fs) = \nabla^1(fs) - \nabla^0(fs) = df \otimes s + f\nabla^1 s - df \otimes s - f\nabla^0 s = fA(s).$$

In other words, A is a tensor:

$$A \in \Gamma^\infty(T^*M \otimes E \otimes E^*) \simeq \Omega^1(M; E \otimes E^*) \simeq \Omega^1(M; \mathrm{End}(E)).$$

Conversely, for any connection ∇^0 and any $A \in \Omega^1(M, \mathrm{End}(E))$ (viewed as a map from $\Gamma^\infty(E)$ to $\Gamma^\infty(T^*M \otimes E)$ as explained above), one can check that $\nabla^0 + A$ is a connection:

$$(\nabla^0 + A)(fs) = \nabla^0(fs) + fA \otimes s = df \otimes s + f\nabla^0 s + fA \otimes s = df \otimes s + f(\nabla^0 + A)s.$$

So

$$\text{The set of linear connections on } E = \nabla^0 + \Omega^1(M; \mathrm{End}(E)).$$

Remark. Given vector bundles with connections, one can perform new connections on new bundles:

- If E is a vector bundle over M and ∇ is a linear connection on E, then one can define a linear connection ∇^* on the dual bundle E^* by requiring
$$d\langle s, s^*\rangle = \langle \nabla s, s^*\rangle + \langle s, \nabla^* s^*\rangle, \qquad \forall s \in \Gamma^\infty(E), s^* \in \Gamma^\infty(E^*),$$
where $\langle\cdot,\cdot\rangle$ is the pairing between E and E^*.
- If E_1, E_2 are vector bundles over M, with two linear connections ∇_1, ∇_2, then one can define the direct sum connection $\nabla = \nabla_1 \oplus \nabla_2$ on $E_1 \oplus E_2$ by requiring
$$\nabla(s_1 \oplus s_2) := \nabla_1 s_1 \oplus \nabla_2 s_2.$$
Similarly we can define the tensor product connection $\nabla = \nabla_1 \otimes \nabla_2$ on the tensor product bundle $E_1 \otimes E_2$ by requiring
$$\nabla(s_1 \otimes s_2) := \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2.$$
As a consequence, any linear connection ∇ on E gives rise to a linear connection on the vector bundle $\mathrm{End}(E)$.

Now let's describe a linear connection ∇ on E locally. Let $\{e_1, \cdots, e_r\}$ be a local frame of E near $x \in M$, i.e. for each y in a neighborhood U of x, $\{e_1(y), \cdots, e_r(y)\}$ form a basis of E_y. Then any section of $E|_U$ can be written as [**In what follows we will apply Einstein's summation convention: automatically sum over repeated upper and lower subscripts**]

$$u = u^j e_j.$$

By definition, one has

$$\nabla u = du^j \otimes e_j + u^j \nabla e_j.$$

So ∇ is completely determined by ∇e_j for a local frame $\{e_1, \cdots, e_r\}$.

Next let's assume that U is a local coordinate patch and the corresponding coordinates near x are given by $\{x^1, \cdots, x^n\}$. Then we get a local frame

$$dx^i \otimes e_j, \quad 1 \le i \le n, 1 \le j \le r$$

of $T^*M \otimes E$. As a consequence, there exist functions Γ_{il}^j on U so that

$$\nabla e_l = \Gamma_{il}^j dx^i \otimes e_j.$$

This implies that for any $u = u^j e_j$,

$$\nabla u = du^j \otimes e_j + \Gamma_{il}^j u^l dx^i \otimes e_j.$$

We let Γ be the following $r \times r$ matrix-valued 1-form (i.e. when paired with a vector, you will get a $r \times r$ matrix)

$$\Gamma = \left(\Gamma_{il}^j dx^i\right)_{1 \le j,l \le r} \in \Omega^1(U) \otimes \mathfrak{gl}(r, \mathbb{R})$$

Then the previous equation can be abbreviated as

$$\nabla u = du + \Gamma u.$$

We will call Γ the *connection 1-form* associated to the given local frame $\{e_1, \cdots, e_r\}$.

Note that the connection 1-form depends on the choice of local frame. Let $\{\tilde{e}_1, \cdots, \tilde{e}_r\}$ be another local frame defined on a coordinate patch U near x. Then we can write u in two ways

$$u^j e_j = u = \tilde{u}^j \tilde{e}_j$$

Let g be the invertible $r \times r$ matrix so that

$$(\tilde{e}_1, \cdots, \tilde{e}_r) = (e_1, \cdots, e_r) g.$$

Then we get

$$\nabla \tilde{e}_l = \widetilde{\Gamma}_{il}^j dx^i \otimes \tilde{e}_j = \widetilde{\Gamma}_{il}^j dx^i \otimes e_s g_j^s = (g_j^s \widetilde{\Gamma}_{il}^j) dx^i \otimes e_s.$$

and

$$\nabla \tilde{e}_l = \nabla(e_j g_l^j) = dg_l^j \otimes e_j + g_l^j \nabla e_l = dg_l^s \otimes e_s + g_l^j \Gamma_{ij}^s dx^i \otimes e_s.$$

Compare the above two formulae we get $g\widetilde{\Gamma} = dg + \Gamma g$, i.e.

$$\widetilde{\Gamma} = g^{-1} dg + g^{-1} \Gamma g.$$

This is the transition rule relating the connection 1-forms in different local frames.

Conversely, one can prove

Proposition 1.3. *Let E be a rank r vector bundle over M and (U_α) an open cover of M consisting of local trivialization charts for E. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ be transition maps of E. Then any collection of matrix-valued 1-forms $\Gamma_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{gl}(r, \mathbb{R})$ satisfying*

$$\Gamma_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} \Gamma_\alpha g_{\alpha\beta}$$

uniquely defines a linear connection on E.

Proof. Exercise. □

2. Curvatures of connections

Now let E be a vector bundle over M and $\nabla : \Omega^0(M;E) \to \Omega^1(M;E)$ a linear connection on E. One can extend ∇ to a family of operators

$$\nabla : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$$

by (Compare: Theorem 2.2 in Lecture 21)

Definition 2.1. Given a linear connection ∇ on E, $\nabla : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$ is given by the formula

$$\begin{aligned}\nabla(\omega \otimes s)(X_0, \cdots, X_k) = &\sum_{i=0}^{k}(-1)^i \nabla(\omega(X_0, \cdots, \widehat{X}_i, \cdots, X_k)s)(X_i) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_0, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_k)s.\end{aligned}$$

One can check

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s, \qquad \forall \omega \in \Omega^k(U), s \in \Gamma^\infty(E).$$

and

$$\nabla(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge \nabla \eta, \qquad \forall \omega \in \Omega^k(M), \eta \in \Omega^l(M;E).$$

Example. Let $E = M \times \mathbb{R}^r$ be the trivial bundle and $\nabla = \nabla^0 : \Omega^0(M;E) \to \Omega^1(M;E)$ be the trivial connection as stated above. Then any element of $\Omega^k(M;E)$ is of the form $\eta = (\eta_1, \cdots, \eta_r)^T$ with $\eta_i \in \Omega^k(M)$, and the map $\nabla : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$ is merely given by

$$\nabla(\eta_1, \cdots, \eta_r)^T = (d\eta_1, \cdots, d\eta_r)^T \in \Omega^{k+1}(M;E).$$

More generally, if $\nabla = d + A$ for some $r \times r$-matrix valued 1-form A, then

$$\nabla(\eta_1, \cdots, \eta_r)^T = (d\eta_1, \cdots, d\eta_r)^T + A \wedge (\eta_1, \cdots, \eta_r)^T \in \Omega^{k+1}(M;E).$$

We have explained that ∇ is not $C^\infty(M)$-linear. It turns out that the composition $\nabla^2 = \nabla \circ \nabla : \Omega^k(M;E) \to \Omega^{k+2}(M;E)$ is $C^\infty(M)$-linear:

Lemma 2.2. *For any $f \in C^\infty(M)$ and any $\omega \in \Omega^k(M;E)$ one has*

$$\nabla^2(f\omega) = f(\nabla^2\omega).$$

Proof. We have

$$\nabla^2(f\omega) = \nabla(df \wedge \omega + f\nabla\omega) = -df \wedge \nabla\omega + df \wedge \nabla\omega + f\nabla^2\omega = f\nabla^2\omega.$$

□

In particular, we see that

$$\nabla^2 : \Omega^0(M;E) = \Gamma^\infty(E) \to \Omega^2(M;E) = \Gamma^\infty(\wedge^2 T^*M \otimes E)$$

is a tensor, and in fact is an $r \times r$ matrix valued 2-form:

$$\nabla^2 \in \Gamma^\infty(\wedge^2 T^*M \otimes E \otimes E^*) \simeq \Omega^2(M; E \otimes E^*) \simeq \Omega^2(M; \mathrm{End}(E)).$$

Note that although the matrix-valued 1-form Γ is only locally defined, the matrix-valued 2-form ∇^2 is globally defined.

Definition 2.3. Given any connection ∇ on E, we will call

$$R_\nabla := \nabla^2 \in \Omega^2(M; \mathrm{End}(E))$$

the *curvature* of ∇.

Example. Again consider the trivial bundle $E = M \times \mathbb{R}^r$. Let $\nabla = d + A$ be any linear connection on E, where A is any $r \times r$ matrix-valued 1-form on M. Then the curvature of ∇ is the two form such that for any $u = (f_1, \cdots, f_r)^T$,

$$R_\nabla u = \nabla(du + Au) = A \wedge du + d(Au) + A \wedge Au = (dA + A \wedge A)u.$$

In other words,

$$R_\nabla = dA + A \wedge A.$$

Note that in the above example, we have $R_{\nabla^0} = 0$.

Definition 2.4. A connection ∇ on E is called *flat* if $R_\nabla = 0$.

Let's do some local computation. Let Γ be the local matrix-valued connection 1-form. In other words, locally after choosing a frame, we have $\nabla = d + \Gamma$. Then by the example above,

$$R_\nabla = d\Gamma + \Gamma \wedge \Gamma.$$

This is called the structure equation. This equation has two consequences:

(1) If we choose a different local frame, then we have seen $\widetilde{\Gamma} = g^{-1}dg + g^{-1}\Gamma g$, where $g : U \to \mathrm{gl}(r, \mathbb{R})$ is the matrix-valued function transforming the frame $(e_1, \cdots, e_r)$ to the new frame $(\tilde{e}_1, \cdots, \tilde{e}_r)$. Then using the fact $dg^{-1} = -g^{-1}(dg)g^{-1}$ (since $d(gg^{-1}) = 0$), we can get

$$\widetilde{R}_\nabla = g^{-1} R_\nabla \; g.$$

(2) If we differentiate both sides of the structure equation, we get

$$dR_\nabla = d\Gamma \wedge \Gamma - \Gamma \wedge d\Gamma = R_\nabla \wedge \Gamma - \Gamma \wedge R_\nabla.$$

This is known as the Bianchi identity. One can prove that if we write $\widetilde{\nabla}$ as the induced linear connection on $\Omega^2(M; \mathrm{End}(E)$, then the Bianchi identity is induced to $\widetilde{\nabla} R_\nabla = 0$.