

**PROBLEM SET 6: DIFFERENTIAL FORMS**  
**DUE DEC. 6**

- (1) Prove Lemma 2.4 and Proposition 2.11 in lecture 20.
- (2) Let  $V, W$  be vector spaces, with basis  $\{v_i\}$  and  $\{w_j\}$  respectively.
- (a) Prove: A nonzero 2-tensor  $T = \sum_{ij} a_{ij} v_i \otimes w_j \in V \otimes W$  is decomposable if and only if the coefficient matrix  $(a_{ij})$  has rank 1.
- (b) Prove:  $V \otimes W^* \simeq L(W, V)$  as vector spaces. Then:
- (i) What is the trace of  $T \in V \otimes V^* \simeq L(V, V)$ ?
- (ii) Why the contraction defined in Definition 1.4 is well-defined?
- (c) **[Universal Property]** Prove: There exists a bilinear map  $\varphi : V \times W \rightarrow V \otimes W$ , so that the following *universal property* holds: for any vector space  $Z$  and any bilinear map  $h : V \times W \rightarrow Z$ , there exists a unique linear map  $\tilde{h} : V \otimes W \rightarrow Z$  so that  $h = \tilde{h} \circ \varphi$ . Moreover, such a pair  $(V \otimes W, \varphi)$  is unique up to isomorphism.
- (3) The famous *Maxwell's equations* in the theory of electromagnetism are of the form

$$\begin{aligned} \nabla \cdot \vec{B} &= 0, & \frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} &= 0, \\ \nabla \cdot \vec{E} &= 4\pi\rho, & \frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{B} &= -4\pi\vec{J}, \end{aligned}$$

where  $\vec{E} = \langle E_x, E_y, E_z \rangle$  is the electricity field,  $\vec{B} = \langle B_x, B_y, B_z \rangle$  is the magnetism field,  $\rho$  is the charge density, and  $\vec{J} = \langle J_x, J_y, J_z \rangle$  is the current density. All functions, e.g.  $E_x, E_y, B_x, \rho, J_x$ , are functions of  $x, y, z$  and  $t$ .

Define the *Faraday 2-form*  $F$  on  $\mathbb{R}^4$  (with coordinates  $t, x, y, z$ ) as

$$\mathbf{F} = E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy,$$

and define the current 1-form to be

$$\mathbf{J} = \rho dt + J_x dx + J_y dy + J_z dz.$$

We also define the *Hodge star operators* on  $\mathbb{R}^4$  (with respect to the *Minkowski metric*) to be the linear operators

$$* : \Omega^k(\mathbb{R}^4) \rightarrow \Omega^{4-k}(\mathbb{R}^4), \quad 0 \leq k \leq 4,$$

so that on 2-forms one has  $** = -1$  and

$$*(dx \wedge dt) = dy \wedge dz, \quad *(dy \wedge dt) = -dx \wedge dz, \quad *(dz \wedge dt) = dx \wedge dy,$$

while on 1-forms one has  $dt \wedge (*dt) = dx \wedge (*dx) = \dots = dt \wedge dx \wedge dy \wedge dz$ , e.g.

$$*dt = dx \wedge dy \wedge dz, \quad *dx = -dt \wedge dy \wedge dz, \quad *dy = dt \wedge dx \wedge dz \quad \text{etc.}$$

Prove: The system of Maxwell equations is equivalent to

$$d\mathbf{F} = 0, \quad d*\mathbf{F} = 4\pi*\mathbf{J}.$$

- (4) Denote by  $(x^1, \dots, x^n, y^1, \dots, y^n)$  the coordinate functions on  $\mathbb{R}^{2n}$ . Let  $\omega = \sum_{i=1}^n dx^i \wedge dy^i$ .
- What is  $d\omega$ ?
  - What is  $\omega^n = \omega \wedge \dots \wedge \omega$  (wedge  $n$  times)?
  - Let  $X = \frac{\partial}{\partial x^1} + \dots + \frac{\partial}{\partial x^k}$ , where  $k \leq n$ . What is  $\iota_X \omega$ ?
  - Let  $\iota : \mathbb{T}^n \hookrightarrow \mathbb{R}^{2n}$  be the submanifold of  $\mathbb{R}^{2n}$  defined by  $(x^i)^2 + (y^i)^2 = 1, 1 \leq i \leq n$ . What is  $\iota^* \omega$ ?

- (5) Prove: **[Cartan's lemma]** Let  $\omega_1, \dots, \omega_k$  be 1-forms on  $U \subset M$  that are linearly independent point-wise. Let  $\eta_1, \dots, \eta_k$  be 1-forms on  $U$  so that  $\sum_i \omega_i \wedge \eta_i = 0$ . Then there exists smooth functions  $A_{ij}$  on  $U$ , satisfying  $A_{ij} = A_{ji}$ , so that  $\eta_i = \sum A_{ij} \omega_j$ .

- (6) Let  $X \in \Gamma^\infty(TM)$  be a smooth vector field on  $M$ . For any  $\omega \in \Omega^k(M)$ , define the *Lie derivative* of  $\omega$  with respect to  $X$  to be

$$\mathcal{L}_X \omega = \left. \frac{d}{dt} \right|_{t=0} \phi_t^* \omega,$$

where  $\phi_t$  is the (local) flow generated by  $X$ . Prove:

- $d\mathcal{L}_X \omega = \mathcal{L}_X d\omega$ .
  - $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta$ .
  - [Cartan's Magic Formula]**  $\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega$ .
  - $(\mathcal{L}_X \omega)(X_1, \dots, X_k) = \mathcal{L}_X(\omega(X_1, \dots, X_k)) - \sum_i \omega(X_1, \dots, \mathcal{L}_X X_i, \dots, X_k)$ .
  - $\mathcal{L}_{[X_1, X_2]} \omega = \mathcal{L}_{X_1} \mathcal{L}_{X_2} \omega - \mathcal{L}_{X_2} \mathcal{L}_{X_1} \omega$ .
- (7)
  - Prove: For any smooth manifold  $M$ , the tangent bundle  $TM$  is orientable.
  - Prove: Any Lie group  $G$  is orientable.
  - Suppose  $M$  is orientable,  $f : M \rightarrow N$  is smooth, and  $q \in N$  is a regular value of  $f$ . Prove:  $f^{-1}(q)$  is orientable.
- (8)
  - Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the map  $f(x) = -x$ . Is  $f$  orientation-preserving?
  - Let  $f : S^n \rightarrow S^n$  be the antipodal map  $f(x) = -x$ . Is  $f$  orientation-preserving?
  - Prove: For any  $k$ ,  $\mathbb{R}P^{2k}$  is not orientable.

- (9) Consider the 2-form  $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$  on  $\mathbb{R}^3$ . Let  $\iota : S^2 \rightarrow \mathbb{R}^3$  be the standard embedding. Compute the integral  $\int_{S^2} \iota^* \omega$ .

- (10) A Riemannian metric on a smooth manifold  $M$  is a symmetric 2-tensor  $g$  so that locally if we write  $g = \sum g_{ij}(x) dx^i \otimes dx^j$ , then the matrix  $(g_{ij}(x))$  is positive definite. Prove the existence of Riemannian metric on any smooth manifold  $M$  using the following two different methods:

- Use partition of unity.
- Use Whitney's embedding theorem.

- (11) **(Not required.)** Read Lee's book *Introduction to Smooth Manifolds* page 493-495 (the subsection on "Involutivity and Differential Forms", and prove Proposition 19.8 on page 495 there.