Real Analysis

Lecture 1
02/26/2018

Instructor Information

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Course Information

- Website: http://staff.ustc.edu.cn/~wangzuoq/Courses/18S-RealAnalysis/index.html

  [Course Notes/Exercises will be uploaded after each lecture.]

- Reference Books:
  - 周民强, 实变函数论(第2版)
  - E. Stein & R. Shakarchi, Real Analysis - measure theory, integration & Hilbert spaces.
  - T. Tao, An introduction to measure theory

- Homeworks:
  - Assigned after class, on the website - Twice every week
  - collected on Mondays, before class

  Exam: One midterm, one final

  Your grades: 30% HW + 30% Midterm + 40% Final

- We plan to cover (may change later)
  - measure theory
  - integration theory] "chicken-egg relation"
  - L^p spaces.

What is Real Analysis + Why do we need it

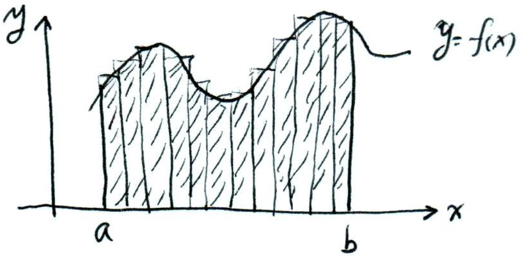
- Wikipedia: Real Analysis deals with real numbers and real-valued functions of a real variable.
  In particular, it deals with the analytic properties of real functions and sequences.
  convergence, continuity, differentiation, integral.

- History.
  17th century (Newton, Leibniz). invent calculus $\longrightarrow$ 18th century (Euler, Laplace, $\cdots$) applications $\longrightarrow$ 19th century (Cauchy, Weierstrass) rigorous: ε-δ $\longrightarrow$ 20th century (Lebesgue, Banach). abstraction.

  In some sense, this course is "real analysis of the 20th century".

- What we learned in mathematical analysis:
  - sequences and series of { real numbers $\leftarrow$ IMPORTANT: $\mathbb{R}$ is complete. ; real-valued functions of real variables }
    $\hookrightarrow$ contains Taylor series, Fourier series etc.
  - differentiation: $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$.
    (slope, rate of change)
  - Riemann integral: $\int_a^b f(x)dx = \lim\limits_{\Delta\to 0} \sum\limits_{i=1}^{n} f(t_i)\,\Delta_i$.
    "length of vertical interval"

    (Area)



idea

  - The fundamental theorem of calculus.
    If $f(x)$ is differentiable in $[a, b]$, and $f'(x)$ is Riemann integrable in $[a, b]$, then $\int_a^b f'(x)\,dx = f(b) - f(a)$.

[Recall. Lebesgue theorem: A bounded function f on $[a, b]$ is Riemann integrable if and only if the set of points where f is discontinuous has measure zero.]

We say a set $S \subset \mathbb{R}$ has measure zero if $\forall \varepsilon > 0$, $\exists$ at most countably many intervals (a_i, b_i) s.t.
(1) $S \subset \bigcup (a_i, b_i)$
(2) $\sum (b_i - a_i) < \varepsilon$.

- Shortcome of Riemann integral

(1) There exist very simple functions (that appears naturally in mathematics) which are NOT Riemann integrable.

Example: $\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$

Then $\chi_{\mathbb{Q}}$ is discontinuous everywhere. By Lebesgue theorem, it is NOT Riemann integrable.

(2) The fundamental theorem of calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

fails if f' is NOT Riemann integrable.

History: Volterra's function (1881): $\exists$ function $f(x)$ whose derivative exists and is bounded everywhere, but f' is not Riemann integrable.

(3) f_n continuous, $f_n \to f$ on $[a,b]$ $\not\Rightarrow$ $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.

One need "uniform convergence" which is a strong assumption.

$\exists$ $0 \le f_n \le 1$, continuous, $f_n(x) \searrow$ as $n \to \infty$, $f_n(x) \to f$ but f is NOT Riem. integrable.

(4). Besicovitch's ~~example~~: $\exists$ a two-variable function f defined on the plane, such that it is Riemann integrable, i.e. $\iint f\,dA$ exists.

But: no matter how do you choose orthogonal coordinate axis x and y, there exists x s.t. $f(x,y)$ is NOT Riemannian integrable w.r.t. y.

In particular, $$\iint f\,dA \neq \int\int f(x,y)\,dy\,dx$$

Note: This is closely related to Kakeya needle problem. (掛谷宗一)

Find a set in $\mathbb{R}^n$ that is as small as possible, so that it contains a unit line segment in every direction

(5). In mordern analysis (like Functional Analysis or Partial Differential Equations), one regard functions as "points", and study properties of various "spaces of functions".

One can define "distance" between two functions, e.g.

$$d(f, g) = \int_a^b |f(x) - g(x)|\, dx.$$

For example, if we let R be the space of Riemann integrable functions on $[a, b]$, then (R, d) is a "metric space".

In particular, one still has the conception of "Cauchy sequence" in such space.

<u>Unfortunately</u>, this space is NOT complete.

[Completeness is crucial for later application. ← Functional Analysis.]

→ need to add "functions" to this space to make it complete.

But these newly added functions are no longer Riemann integrable.

(6). For any Riemann integrable function f on $[-\pi, \pi]$, one has its Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

where $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$. We know

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2\, dx \qquad \text{(Parseval's identity)}$$

Question: what coefficients a_n can occur as the Fourier coefficients?

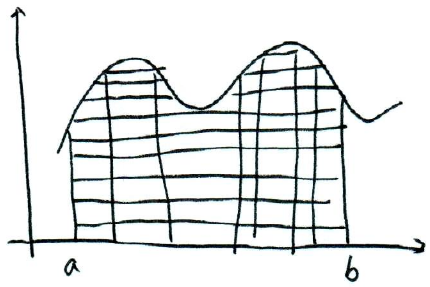
instead of assuming $\sum |a_n| < \infty$, can we simply assume $\sum |a_n|^2 < \infty$?

This is one of the original motivation for the theory of Lebesgue measure/integration.

It turns out the set of points where $\sum a_n e^{inx}$ converges is quite complicated, and even if it converges, to some function f, generally it will not be Riemann integrable.

<u>Conclusion</u>: We need a new theory of integration!

Rough idea of Lebesgue's theory



For positive "nice" functions:

"$\int f(x)\,dx = \int_0^\infty m(\{x : f(x) > t\})\,dt$."

m: measure = "size of set".

compare to Riemann integral:

"different ways to count coins".

It turns out that this new integral theory of Lebesgue overcome the shortcome of Riemann integral that we just listed:

(1) One can now integrate more functions.

In particular, $\chi_{\mathbb{Q}}$ is Lebesgue integrable (since $\mathbb{Q}$ is of measure zero), and

$$\int \chi_{\mathbb{Q}}(x)\,dx = 0.$$

(2) The fundamental theorem of calculus holds for much wider class of functions ("absolute continuous")

(3). Simple criteria for convergence of integral from convergence of functions.

"Lebesgue dominated convergence theorem",

(4) Fubini theorem: "double integral = iterated integral."

holds for a very wide class of functions.

(5) The space of functions whose p-th ($p \geq 1$) power is Lebesgue integrable is complete w.r.t. the ~~metric~~ metric

$$d_p(f, g) = \left(\int |f(x) - g(x)|^p\,dx\right)^{1/p}$$

This is the space L^p.

(6) Plancherel's Theorem: $\sum |a_n|^2 < \infty \Rightarrow \exists!\ f \in L^2([-\pi, \pi])$ s.t. the Fourier coeff's of f are a_n's.

<u>Moreover</u>: the way to construct Lebesgue measure/integral shed a light on defining $\underbrace{\text{measures/integrals}}_{\text{general/abstract}}$ one abstract spaces, which is very useful in modern mathematics

$\boxed{\text{Key conceptions of Lebesgue theory}}$

- <u>measure</u> of a set = "size" of the set.

  e.g. measure of an interval = length of the interval

  ------ rectangle = area ---- rectangle

  --------- solid = volume --- solid

- Let's concentrate on subsets E of $\mathbb{R}$.

  A measure should be a function $m: \{\text{subsets of } \mathbb{R}\} \to [0, \infty]$ such that

  (1) $m([a,b]) = b-a$

  (2) If $\{E_i\}$ are subsets of $\mathbb{R}$ s.t. $E_i \cap E_j = \phi$, $\forall i \neq j$, then

  $$m(\bigcup_i E_i) = \sum m(E_i).$$

  [countable or finite].

  (3) For any $x \in \mathbb{R}$, one has ← (Important for Lebesgue's measure)

  $$m(E+x) = m(E).$$

  <u>Unfortunately</u>: There exists no "measure" if you want to measure all sets.

  (require "axiom of choice").

  <u>Reason</u>: Define an equivalente relation $\sim$ on $[0,1)$ by

  $$x \sim y \Leftrightarrow x - y \in \mathbb{Q}.$$

  Define a set $E = \{\text{one number from each equivalence class}\} \subset [0,1)$. ← choose $0 \in E$

  For each $r \in \mathbb{Q}^{[0,1)}$, let $E_r = \{x + r \ (\text{mod } \mathbb{Z}) : x \in E\} \subset [0,1)$.

  Then all these E_r's are distinct.

  Suppose such a measure exists, then $m(E_r) = m(E)$, $\forall r$.

  But $[0,1) = \bigcup_r E_r$. $\Rightarrow 1 = m([0,1)) = \sum_r m(E_r) = \infty \cdot m(E_r)$

  $\Rightarrow m(E_r) = 0$, $\forall r$

  $\Rightarrow m([0,1)) = 0$, contradiction.

  (Alternatively, regard $[0,1)$ as S^1. Do a similar construction.)

<u>Remark</u>: Even if we only allow "finite additivity": $m(\bigcup_{i=1}^N E_i) = \sum_{i=1}^N m(E_i)$ for $E_i \cap E_j = \phi$.

one still can't measure all subsets of $\mathbb{R}^n$.

⇝ <u>Banach-Tarski paradox</u>: One can "cut" a unit ball $B = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$ into five pieces, and after translating + rotating each of these pieces, reassemble them into two disjoint unit balls!

In conclusion: we can't measure all subsets.
We can only measure some subsets. $\rightsquigarrow$ <u>measurable sets</u>.

- In view of the formula in defining Lebesgue integral of positive functions,

$$\int f(x)\,dx = \int_0^\infty m(\{x : f(x) > t\})\,dt.$$

we can only integrate functions f s.t. the sets $\{x : f(x) > t\}$ are measurable.
(for <u>almost every</u> t). $\rightsquigarrow$ <u>measurable functions</u>

$\uparrow$

$\boxed{\text{Very important conception}}$ $\rightsquigarrow$ <u>Lebesgue integrable functions</u>

- <u>A warning</u>: To understand the subject, people had constructed many many weird examples.

  <u>BUT</u>: REAL ANALYSIS IS NOT A SUBJECT OF WEIRD EXAMPLES.

  It is a theory that tells you to what extend many nice properties still holds.
  For another example: It is in the framework of Lebesgue's theory that one can see that any convex function on (a, b) is a.e. differentiable (and in fact twice differentiable a.e.)

- In fact, new theory is NOT very far from the theory that we are familiar with.
  "<u>Littlewood's three principles</u>"
  - Every measurable set is <u>nearly</u> a finite union of intervals
  - Every measurable function is <u>nearly</u> continuous
  - Every convergence sequence of measurable functions is <u>nearly</u> uniformly convergence.

  We will see exact statement of these principles later in the course.