Real Analysis

03/01/2018
Lecture 2

Last time

- We need a theory that supersedes Riemann integral.
- Lebesgue's idea: measure the size of $\boxed{\text{more complicated}}$ sets that intervals.
- One can not measure all subsets of $\mathbb{R}$.

Today: Elementary measure, Jordan measure

## 1. Elementary measure

- Notation: $I = [a, b]$ or $[a, b)$ or $(a, b]$ or $(a, b)$ interval.

  (We allow $\{a\} = [a, a]$ and $\phi = (a, a)$)

  $\rightsquigarrow$ length $|I| = m(I) = b - a$.

- $B = I_1 \times \cdots \times I_d = (a_1, b_1) \times \cdots \times (a_d, b_d) \subset \mathbb{R}^d$ box.

  $\rightsquigarrow$ volume $|B| = m(B) = |I_1| \cdot \cdots \cdot |I_d|$

**Def.** A set $E \subset \mathbb{R}^d$ is an elementary set if it is the union of finitely many boxes, i.e.

$$E = B_1 \cup B_2 \cup \cdots \cup B_n. \quad \text{(may intersect)}.$$

One can check that the union, intersection and set-theoretic difference of elementary sets are still elementary sets.

We will denote the set of all elementary sets in $\mathbb{R}^d$ by $\mathcal{E}$

- **Prop.** Any elementary set $E$ can be expressed as the union of finitely many pairwise disjoint boxes.

**Proof.** ~~The statement is obviously true for d=1, since the union of two intervals is either the union of disjoint intervals, or it is one larger interval. (Arrange intervals in the order by looking at their left-most point.)~~

~~For $d>1$, suppose $E = B_1 \cup B_2 \cup \cdots \cup B_n$, with $B_i = I_{i,1} \times \cdots \times I_{i,d}$.~~

- For $d=1$, i.e. $E = I_1 \cup \cdots \cup I_n$, we rearrange all endpoints of $I_j$'s in an increasing order as $c_1, c_2, \cdots, c_N$. Let $I_1' = [c_1, c_1]$, $I_2' = (c_1, c_2)$, $I_3' = [c_2, c_2]$, $I_4' = (c_2, c_3)$, $\cdots$, $I_{2N-1}' = [c_N, c_N]$.

  Pick those $I_j'$ that lie in $E$. Then they are disjoint and their union is $E$.
- For $d>1$, suppose $E = B_1 \cup B_2 \cup \cdots \cup B_n$, where $B_i = I_{i,1} \times \cdots \times I_{i,d}$.

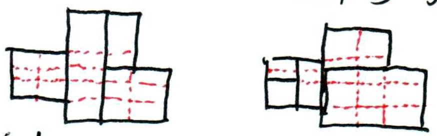
  For each $1 \le l \le d$, apply the 1-dim argument above to $I_{1,l}, I_{2,l}, \cdots, I_{n,l}$.

  Then we can write each $B_i$ (and also $E$) as the finite union of disjoint small boxes. $\square$

• **Def.** Let $E = B_1 \cup \cdots \cup B_n$ be an elementary set, s.t. $B_i \cap B_j = \phi$, $\forall i,j$.
Then we define the <u>measure</u> of $E$ to be
$$m(E) = m(B_1) + m(B_2) + \cdots + m(B_n).$$

One need to justify that the definition is reasonable, i.e. $m(E)$ is independent of different partitions of $E$ into disjoint boxes. There are two different ways to argue this.

<u>Method 1</u>. Find a mutual refinement of two given partitions.
the method is similar to the proof of the above proposition.



$\leftarrow$ idea.

4 boxes $\rightarrow$ 19 boxes $\leftarrow$ 5 boxes

<u>Method 2</u>. Prove an explicit formula for $m(E)$.
$$m(E) = \lim_{N\to\infty} \frac{1}{N^d} \#\left(E \cap \frac{1}{N}\mathbb{Z}^d\right). \qquad \text{(discretization formula)}$$
We will leave this as an exercise.

• By this way, we get a reasonable definition of "measure" for all elementary sets. The following properties are easy to check from the definition.

<u>Prop.</u> The "measure" $m: \mathcal{E} \to [0, +\infty)$ satisfies
(1) If $E_1, E_2$ are disjoint elementary sets, then $m(E_1 \cup E_2) = m(E_1) + m(E_2)$ $\leftarrow$ (finite additivity)
(2) If $B$ is a box, then $m(B) = |B|$.
(3) If $E_1 \subset E_2$ are elementary, then $m(E_1) \le m(E_2)$. $\leftarrow$ (monotonicity)
(4) For any two elementary sets $E_1, E_2$, we have $m(E_1 \cup E_2) \le m(E_1) + m(E_2)$ $\leftarrow$ (sub-additivity)
(5) For any $x \in \mathbb{R}^n$ and elementary set $E$, $m(E + x) = m(E)$. $\leftarrow$ (translation invariance)

In fact, the elementary measure on elementary sets is characterized by finite additivity, translation-invariance and a normalization condition.

<u>Thm.</u> Suppose $\tilde{m}: \mathcal{E} \to [0, +\infty)$ is finite-additive, translation-invariant, and $\tilde{m}([0,1]^d) = 1$.
Then $\tilde{m} = m$.

<u>"sketch of Proof"</u>. • Let $E = B_1 \cup \cdots \cup B_n$ be a disjoint union, then by finite-additivity,
$$\tilde{m}(E) = \tilde{m}(B_1) + \cdots + \tilde{m}(B_n), \qquad m(E) = m(B_1) + \cdots + m(B_n).$$
To prove the theorem, it is enough to prove $\tilde{m}(B) = m(B)$ for any box.

• Since $[0,1) = [0, \frac{1}{k}) \cup [\frac{1}{k}, \frac{2}{k}) \cup \cdots \cup [\frac{k-1}{k}, 1)$, one can apply finite additivity and translation invariance to prove $\tilde{m}([0, \frac{1}{k})^d) = (\frac{1}{k})^d \cdot \tilde{m}([0,1]^d) = (\frac{1}{k})^d$
and in general $\tilde{m}([a_1, b_1) \times \cdots \times [a_d, b_d)) = (b_1 - a_1) \times \cdots \times (b_d - a_d)$ for $a_i, b_i \in \mathbb{Q}$.

• Since $\tilde{m}$ is non-negative and finite-additive, it is monotone.
Now for general $B = I_1 \times \cdots \times I_d$, one can construct boxes $B^-, B^+$ with <u>rational endpoints</u> s.t. $B^- \subset B \subset B^+$, and "$\varepsilon$-close" to $B$. Finally take limit. $\square$

2.

## 2. Jordan measure

- Now we have a nice theory of measure for all elementary sets. But, how do we measure triangles? disks?

Archimedes' observation: They can be approximated from both inside and outside by elementary sets.

(Moreover, both approximation give the same limit)

**Def:** Let $A \subset \mathbb{R}^d$ be a bounded set.

(1) The Jordan inner measure is $J_*(A) := \sup_{\mathcal{E} \ni E \subset A} m(E)$

(2) The Jordan outer measure is $J^*(A) := \inf_{\mathcal{E} \ni F \supset A} m(F)$.

(3) We say $A$ is Jordan measurable if $J_*(A) = J^*(A)$.

The Jordan measure of such a set $A$ is $J(A) = J_*(A) = J^*(A)$

Note:
① $J_*$, $J^*$ are defined for all sets in $\mathbb{R}^d$
② $J_* \le J^*$

- Obviously if $E$ is elementary, then it is Jordan measurable, and $m(E) = J(E) = J_*(E) = J^*(E)$

The next theorem says that any Jordan measurable set is "nearly elementary".

**Thm:** Let $A \subset \mathbb{R}^d$ be any bounded set. Then T.F.A.E.

(1) $A$ is Jordan measurable.

(2) $\forall \varepsilon > 0$, $\exists E, F \in \mathcal{E}$ s.t. $E \subset A \subset F$ and $m(F \setminus E) < \varepsilon$.

(3) $\forall \varepsilon > 0$, $\exists E \in \mathcal{E}$ s.t. $E \subset A$ and $J^*(A \setminus E) < \varepsilon$.

(4) $\forall \varepsilon > 0$, $\exists F \in \mathcal{E}$ s.t. $A \subset F$ and $J^*(F \setminus A) < \varepsilon$.

Example:
$A_1 = [0,1] \cap \mathbb{Q}$,
$A_2 = [0,1] \setminus A_1$.
Then
$J_*(A_1) = J_*(A_2) = 0$
$J^*(A_1) = J^*(A_2) = 1$.
$\Downarrow$
- $J_*$ is NOT subadditive
- $J^*$ is NOT finite additive

Before we prove the theorem, we prove three simple lemmas.

**Lemma 1.** Suppose $A_1 \subset A_2$ are bounded, then $J_*(A_1) \le J_*(A_2)$, $J^*(A_1) \le J^*(A_2)$

*Proof:* let $E_n \subset A_1 \subseteq A_2 \subset F_n$ be sequences of elementary sets s.t.

$$m(E_n) \to J_*(A_1), \qquad m(F_n) \to J^*(A_2).$$

Then

$$J_*(A_2) \ge m(E_n) \to J_*(A_1)$$
$$J^*(A_1) \le m(F_n) \to J^*(A_2). \qquad \square$$

**Lemma 2:** Suppose $A_1, A_2$ are bounded. Then $J^*(A_1 \cup A_2) \le J^*(A_1) + J^*(A_2)$

*Proof.* let $F_n, G_n$ be sequences of elementary sets, $A_1 \subset F_n$, $A_2 \subset G_n$, s.t.

$$m(F_n) \to J^*(A_1), \qquad m(G_n) \to J^*(A_2).$$

Then $A_1 \cup A_2 \subset F_n \cup G_n$, and thus

$$J^*(A_1 \cup A_2) \le m(F_n \cup G_n) \le m(F_n) + m(G_n) \to J^*(A_1) + J^*(A_2). \qquad \square$$

**Lemma 3.** Suppose $A \subset B$, where $B$ is a box. Then $A$ is Jordan measurable iff $B \setminus A$ is Jordan measurable.

*Proof:* By def., $J_*(B \setminus A) = m(B) - J^*(A)$, $J^*(B \setminus A) = m(B) - J_*(A)$

So $J^*(A) = J_*(A) \Leftrightarrow J_*(B \setminus A) = J^*(B \setminus A)$. $\square$

## Proof of the theorem.

(1) ⇒ (2): Suppose $A$ is Jordan measurable.

Then $\forall \varepsilon > 0$, $\exists$ elementary sets $E \subset A \subset F$ s.t. $m(F) - \frac{\varepsilon}{2} < J^*(A) = J(A) = J_*(A) < m(E) + \frac{\varepsilon}{2}$.

This implies (since $F = E \cup (F \setminus E)$ is a disjoint union)

$$m(F\setminus E) + m(E) = m(F) \le m(E) + \varepsilon$$

So $m(F\setminus E) < \varepsilon$.

(2) ⇒ (3), (4): Suppose (2) holds. Then $A \setminus E \subset F \setminus E$, $F \setminus A \subset F \setminus E$.

By lemma 1, we have

$$J^*(A\setminus E) \le J^*(F\setminus E) = m(F\setminus E) < \varepsilon,$$
$$J^*(F\setminus A) \le J^*(F\setminus E) = m(F\setminus E) < \varepsilon.$$

(3) ⇒ (1): $\forall \varepsilon > 0$, $\exists E_\varepsilon \in \mathcal{E}$ s.t. $E_\varepsilon \subset A$ and $J^*(A\setminus E_\varepsilon) < \varepsilon$.

Since $E_\varepsilon$ is elementary, we have $m(E_\varepsilon) \le J_*(A)$.

Thus by lemma 2, we get

$$J^*(A) \le J^*(A\setminus E_\varepsilon) + J^*(E_\varepsilon) < \varepsilon + m(E_\varepsilon) \le \varepsilon + J_*(A).$$

Taking $\varepsilon \to 0$, we get $J^*(A) \le J_*(A)$. So $J^*(A) = J_*(A)$, i.e. $A$ is Jordan measurable.

(4) ⇒ (1): $\forall \varepsilon > 0$, $\exists F_\varepsilon \in \mathcal{E}$ s.t. $A \subset F_\varepsilon$ and $J^*(F_\varepsilon \setminus A) < \varepsilon$.

Take a box $B$ s.t. $A \subset B$. We may assume $F_\varepsilon \subset B$ (otherwise, replace $F_\varepsilon$ by $F_\varepsilon \cap B$)

Let $E_\varepsilon = B \setminus F_\varepsilon \subset B\setminus A$. Then

$$J^*((B\setminus A)\setminus E_\varepsilon) = J^*(F_\varepsilon \setminus A) < \varepsilon.$$

By last step, we conclude that $B \setminus A$ is Jordan measurable.

By lemma 3, $A$ is Jordan measurable. □

~~We end with several remarks.~~

- As for the elementary measure, one can prove that the Jordan measure

$$J : \mathcal{J} \to [0, +\infty)$$

satisfies (1) If $A, B \in \mathcal{J}$ are disjoint, then $J(A\cup B) = J(A) + J(B)$ (finite additivity)

(2) If $A \subset B$, are Jordan measurable, then $J(A) \le J(B)$ (monotonicity)

(3) For $A, B \in \mathcal{J}$, $J(A\cup B) \le J(A) + J(B)$ (subadditivity)

(4) For $A \in \mathcal{J}$, $J(A + x) = J(A)$ (translation-invariance)

Conversely, Jordan measure is the only finite-additive, translation-invariant measure on $\mathcal{J}$ with $J([0,1]^d) = 1$

- Jordan measure is closely related to the Riemann integral
  - A bounded set $A \subset \mathbb{R}^d$ is Jordan measurable if and only if $\chi_A$ is Riemann integrable.
  - A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if $E_+, E_-$ are Jordan measurable, in which case one has $\int_a^b f(x)\,dx = J(E_+) - J(E_-)$.